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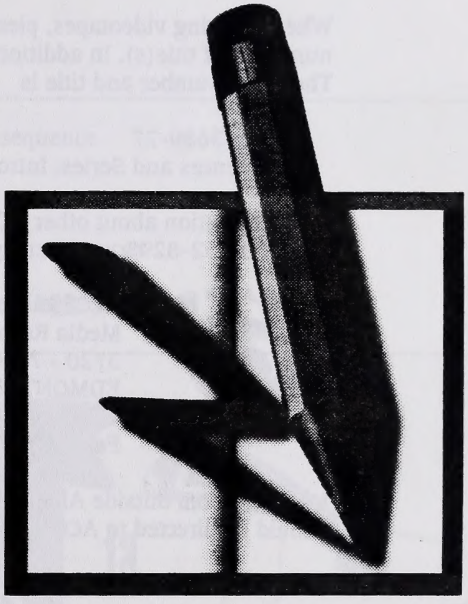

Lesson 1:
Introduction to Sequences

Student Workbook

Produced by

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This guide accompanies **Introduction to Sequences** from the Math Factor series.

Writer: Jim Ashton
Editor: Wilma Gonsalves
Producer: Katherine Rankin

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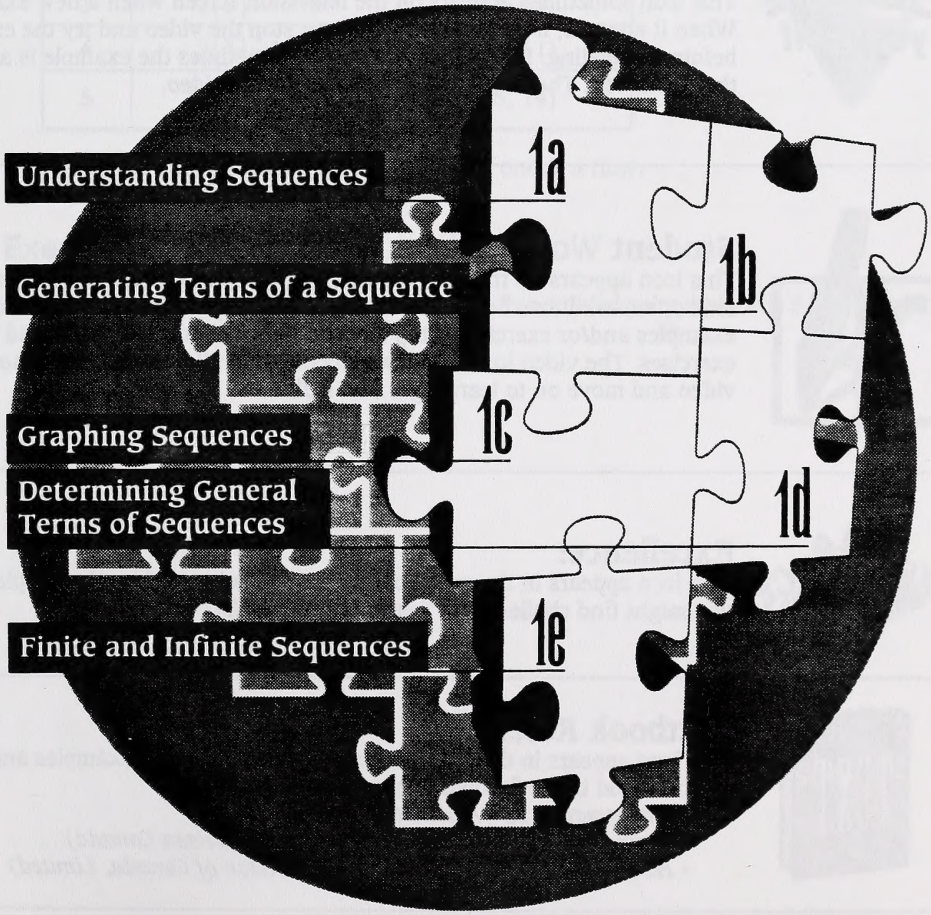
Lesson 1: INTRODUCTION TO SEQUENCES

LEARNER EXPECTATIONS

By the end of Lesson 1 you should be able to

- understand the general concept of a sequence
- generate the terms of a sequence
- differentiate between finite and infinite sequences
- determine the functions that describe simple sequences

This lesson has five parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

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- challenge yourself to find the solution before it is given
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- *Mathematics 12 (Addison-Wesley)*
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- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Building Blocks

Do you remember how to use a relation to determine ordered pairs of numbers? Before watching this program, refresh your memory by studying the following example and doing the exercises.

Example

For the relation $y = 3x - 1$, determine the four ordered pairs when x is 2, 3, 4 and 5.

Solution

Use a table of values.

x	$y = 3x - 1$	(x, y)
2	$3(2) - 1 = 5$	(2, 5)
3	$3(3) - 1 = 8$	(3, 8)
4	$3(4) - 1 = 11$	(4, 11)
5	$3(5) - 1 = 14$	(5, 14)

The values of x are substituted in the function one at a time.

Exercises

1. For the relation $y = 2x - 4$, determine the five ordered pairs when x is 5, 6, 7, 8 and 9.

2. For the relation $y = x^2 + 3$, determine the three ordered pairs when x is 1, 2 and 3.

BUILDING BLOCKS SOLUTIONS are at the back of this workbook.

Understanding Sequences

Find the next three terms: $-1, 3, -9, 27, \_\_\_, \_\_\_, \_\_\_$.

Watch the Introduction and Video Segment 1a.

Example 1

Find the next three terms in the sequence
$-1, -4, -9, -16, \_\_\_, \_\_\_, \_\_\_$.

Hints for Predicting Terms

- Is there a common difference between the terms?
- Is there a common ratio or multiplier between the terms?
- Is the difference between them increasing at a fixed rate?
- Is each term related to the two previous terms?
- Is each term related to a power of a number?

Solution

The differences are as follows:
$-3, -5, -7$

Note that the differences form a decreasing pattern. The next three differences are -9, -11 and -13. Therefore, this gives the result
$-1, -4, -9, -16, -25, -36, -49$.

An alternate way to arrive at the solution is to observe that each term is the negative of the squares of consecutive integers.

Example 2

Find the next three terms in the following sequence:

$2, 8, 26, 80, \_\_\_, \_\_\_, \_\_\_$

Solution

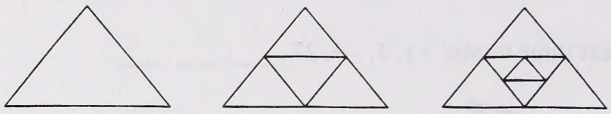
Note that each term is one less than a power of 3. For example, 80 is $3^4 - 1$. Following this pattern, the sequence is as follows:
$2, 8, 26, 80, 242, 728, 2186$

An alternate way to arrive at the solution is to observe that each term is 2 more than 3 times the previous term.

Additional Example
Mathematics 12: p. 371 (Example 1)

Exercises

1. Draw the next two diagrams that will continue this pattern:



2. Find the next three terms for each of the following:

   a. the geometric sequence 16, 64, 256, ___, ___, ___

   b. the arithmetic sequence −1, 1, 3, ___, ___, ___

   c. the sequence 3, 5, 9, 17, ___, ___, ___

   d. the geometric sequence $\sqrt{7}$, 7, $7\sqrt{7}$, 49, ___, ___, ___

EXERCISE 1a SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 372 (Questions 2, 3)
Mathematics: Principles and Process 12: p. 352 (Questions 6, 10), p. 351 (Question 4)
Holtmath 12: p. 417 (Question 1), p. 420 (Question 8)

Generating Terms of a Sequence

Challenge yourself

Find the first five terms of the sequence that has the general term $t_n = n^2 + 5$.

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Watch Video Segment 1b.

Example

List the first four terms, the domain and the range for the sequence whose nth term, t_n, is $n^3 - 1$.

Solution

Recall that t_n can be written as $f(n)$.

$n = 1 \quad t_1 = 1^3 - 1 = 0$

$n = 2 \quad t_2 = 2^3 - 1 = 7$

$n = 3 \quad t_3 = 3^3 - 1 = 26$

$n = 4 \quad t_4 = 4^3 - 1 = 63$

N is the set of natural numbers (1, 2, 3, 4, ...). The domain of a sequence will always be the set of natural numbers or a subset of the set of natural numbers.

The domain is {1, 2, 3, 4, ...} or $n \in \mathrm{N}$.

The range is {0, 7, 26, 63, ...}.

Note: These are the terms of the sequence.

TEXT

Additional Examples
Mathematics 12: p. 371 (Example 2)
Mathematics: Principles and Process 12: p. 350
Holtmath 12: p. 418

Exercises

3. List the first five terms of the sequence defined by the function $f(n)$, where $f(n) = t_n$ and $t_n = 2n^2 - n$.

4. Find t_{12} for the sequence that has the general term $t_n = -4n + 15$.

5. For the sequence $10, 8, 6, 4, \ldots, -8$, determine the following:

   a. the number of terms

   b. the domain and range

   c. an expression for t_n, the general term for the sequence

EXERCISE 1b SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 372 (Questions 4, 6, 7)
Mathematics: Principles and Process 12: p. 351 (Question 1), p. 352 (Question 6)
Holtmath 12: p. 420 (Questions 5, 7)

Graphing Sequences

Watch Video Segment 1c.

Example

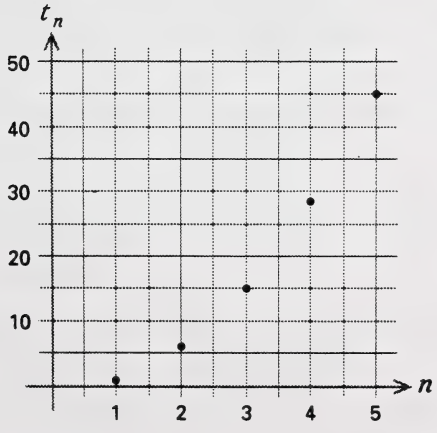
Sketch the graph for the first five terms of the sequence where the general term is $t_n = 2n^2 - n$.

Solution

The following table shows the values of t_n for each value of n:

n	t_n
1	1
2	6
3	15
4	28
5	45

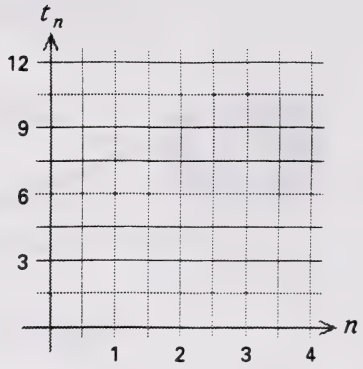
Use these values as ordered pairs for the graph.



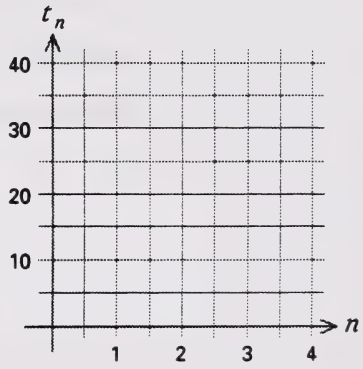
The dots are **not** connected since the domain contains only the natural numbers.

Exercises

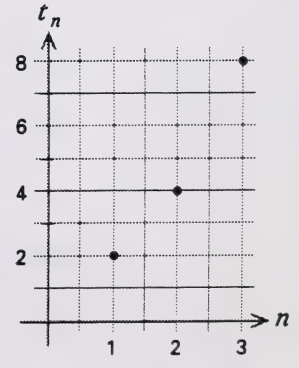
6. Sketch the graph of the first four terms of the sequence that has the general term $t_n = 2n + 3$.



7. Sketch the graph of the first four terms of the sequence that has the general term $t_n = 2^{n+1}$.



8. Answer the following using the given graph of a sequence.



   a. What is the value of t_3?

   b. What is the value of t_4?

   c. Give an expression for t_n.

EXERCISE 1c SOLUTIONS are at the back of this workbook.

Determining General Terms of Sequences

Find the general term of the sequence $1, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \ldots$.

Watch Video Segment 1d.

To obtain general terms of sequences, you must first determine what operations are happening between the consecutive terms of the sequence. Look at the following two examples.

- 7, 10, 13, 16, ...
  In this sequence, subtracting any two consecutive terms yields a common difference of 3.

- 3, 6, 12, 24, ...
  In this sequence, dividing any two consecutive terms yields a common ratio of 2.

The following table illustrates how to find general terms for various cases. For the first two cases, you will be given a formula in a later lesson. In the other cases, the following ideas, along with a little trial and error, will help you.

Between Terms	Hint for General Term	Example	Comment	$t_n =$
common difference	Multiply n by the difference.	1, 3, 5, 7, ...	The common difference is 2.	$t_n = 2n - 1$
common ratio or multiplier	Raise the common ratio to some power.	4, 8, 16, 32, ...	The common ratio is 2.	$t_n = 2^{n+1}$
difference increases at a fixed rate	Use n^2.	1, 4, 9, 16, 25, ...	The differences of 3, 5, 7, etc., are increasing at a fixed rate.	$t_n = n^2$
difference increases at a variable rate	Use the higher power of n.	1, 8, 27, 64, ...	The differences of 7, 19, 37 have a variable increase.	$t_n = n^3$

Additional Example
Mathematics 12: p. 371 (Example 3)

Exercises

9. Determine the general term for each of the following sequences.

   a. a geometric sequence 3, 9, 27, ...

   b. an arithmetic sequence 7, 10, 13, 16, ...

   c. 0, 3, 8, 15, 24, ...

   d. 2, 9, 28, 65, ...

10. Determine the general term for the sequence $1, \frac{2}{5}, \frac{1}{3}, \frac{4}{13}, \frac{5}{17}, \ldots$.

EXERCISE 1d SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 372 (Questions 5, 8, 9)
Mathematics: Principles and Process 12: p. 351 (Question 4), p. 352 (Question 9)
Holtmath 12: p. 420 (Question 9)

Finite and Infinite Sequences

Watch Video Segment 1e.

Example

The first three terms of a sequence are 1, 3, 9, ... Using this information, answer the following:

- Is this sequence infinite or finite?
- Write this sequence three different ways.

Solution

This sequence is finite since it has a limited number of terms.

Three ways this sequence can be written are as follows:

- 1, 3, 9, 27, 81, 243, 729
- 1, 3, 9, ..., 729
- $t_n = 3^{n-1}$, $1 \leq n \leq 7$, $n \in \mathrm{N}$

Additional Example
Mathematics: Principles and Process 12: p. 350

Exercises

11. Given the general term $t_n = 2n - 1$, $n \in \mathrm{N}$, write the infinite sequence it represents in three different ways.

12. Determine whether the following sequences are finite or infinite.

a. 1, 2, 4, ..., 1024

b. 1, 3, 6, ..., t_n, ...

c. $t_n = 2^n - 1,\ n \in \mathrm{N}$

d. $t_n = 3n + 2,\ 1 \le n \le 5,\ n \in \mathrm{N}$

EXERCISE 1e SOLUTIONS are at the back of this workbook.

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. In the next lesson, you will learn how to define a sequence recursively – that is, by the previous term or terms.

Solutions

BUILDING BLOCKS

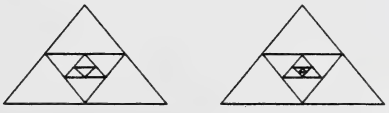
1.

x	$y = 2x - 4$	(x, y)
5	$2(5) - 4 = 6$	(5, 6)
6	$2(6) - 4 = 8$	(6, 8)
7	$2(7) - 4 = 10$	(7, 10)
8	$2(8) - 4 = 12$	(8, 12)
9	$2(9) - 4 = 14$	(9, 14)

2.

x	$y = x^2 + 3$	(x, y)
1	$(1)^2 + 3 = 4$	(1, 4)
2	$(2)^2 + 3 = 7$	(2, 7)
3	$(3)^2 + 3 = 12$	(3, 12)

EXERCISE 1a

1. 

2. a. 16, 64, 256, **1024, 4096, 16 384**
(Multiply by 4.)

b. -1, 1, 3, **5**, **7**, **9**
(Add 2.)

c. 3, 5, 9, 17, **33**, **65**, **129**
(Add double the amount each time.)

d. $\sqrt{7}$, 7, $7\sqrt{7}$, 49, $\mathbf{49\sqrt{7}}$, **343**, $\mathbf{343\sqrt{7}}$
(Multiply by $\sqrt{7}$.)

EXERCISE 1b

3. $n = 1$ $\quad t_1 = 2(1)^2 - 1 = 1$

$n = 2$ $\quad t_2 = 2(2)^2 - 2 = 6$

$n = 3$ $\quad t_3 = 2(3)^2 - 3 = 15$

$n = 4$ $\quad t_4 = 2(4)^2 - 4 = 28$

$n = 5$ $\quad t_5 = 2(5)^2 - 5 = 45$

4. $t_{12} = -4(12) + 15 = -33$

5. a. The sequence 10, 8, 6, 4, 2, 0, -2, -4, -6, -8 has ten terms.

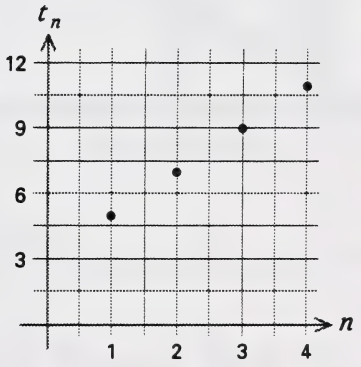
b. domain: $\{1 \le n \le 10, n \in \mathrm{N}\}$
range: $\{10, 8, 6, 4, 2, 0, -2, -4, -6, -8\}$

c. $t_n = 12 - 2n$ or $t_n = 10 + (-2)(n - 1)$

EXERCISE 1c

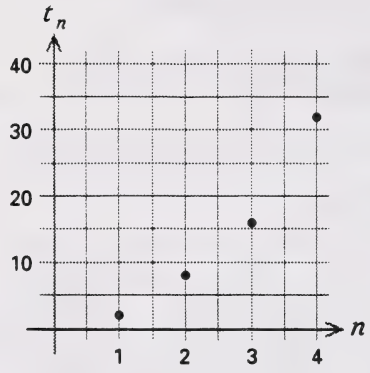
6.

n	t_n	(n, t_n)
1	$2(1) + 3 = 5$	$(1, 5)$
2	$2(2) + 3 = 7$	$(2, 7)$
3	$2(3) + 3 = 9$	$(3, 9)$
4	$2(4) + 3 = 11$	$(4, 11)$



7.

n	t_n	(n, t_n)
1	$2^{1+1} = 4$	$(1, 4)$
2	$2^{2+1} = 8$	$(2, 8)$
3	$2^{3+1} = 16$	$(3, 16)$
4	$2^{4+1} = 32$	$(4, 32)$



8. a. $t_3 = 8$

b. $t_1 = 2, t_2 = 4, t_3 = 8, t_4 = 16$

c. $t_n = 2^n$

EXERCISE 1d

9. a. There is a common ratio of 3.

$t_n = 3^n$

b. There is a common difference of 3.

$t_n = 3n + 4$

c. The differences are increasing at a fixed rate.

$t_n = n^2 - 1$

d. The differences are increasing at a variable rate.

$t_n = n^3 + 1$

10. Express $\frac{1}{3}$ as $\frac{3}{9}$.

$1, \frac{2}{5}, \frac{3}{9}, \frac{4}{13}, \frac{5}{17}$

Now you can see the numerator is increasing by 1 and the denominator is increasing by 4.

$t_n = \frac{n}{4n-3}$

EXERCISE 1e

11. 1, 3, 5, 7, 9, ...

1, 3, 5, ..., $2n - 1$, ...

1, 3, 5, ..., t_n, ...

12. a. finite

b. infinite

c. infinite

d. finite

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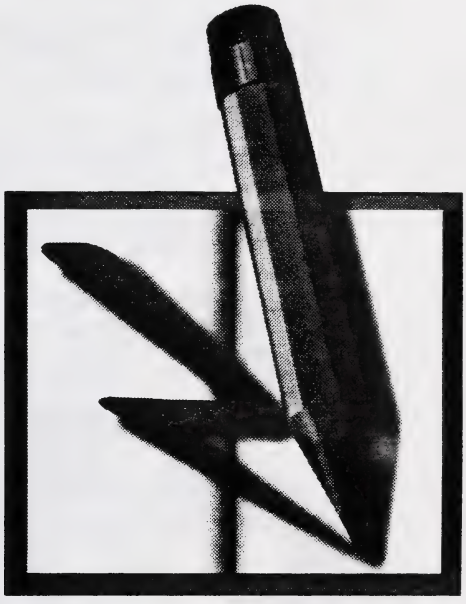

Lesson 2:
Sequences: Recursive Forms

Student Workbook

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This guide accompanies **Sequences: Recursive Forms** from the Math Factor series.

Writer: Jim Ashton
Editor: Wilma Gonsalves
Producer: Katherine Rankin

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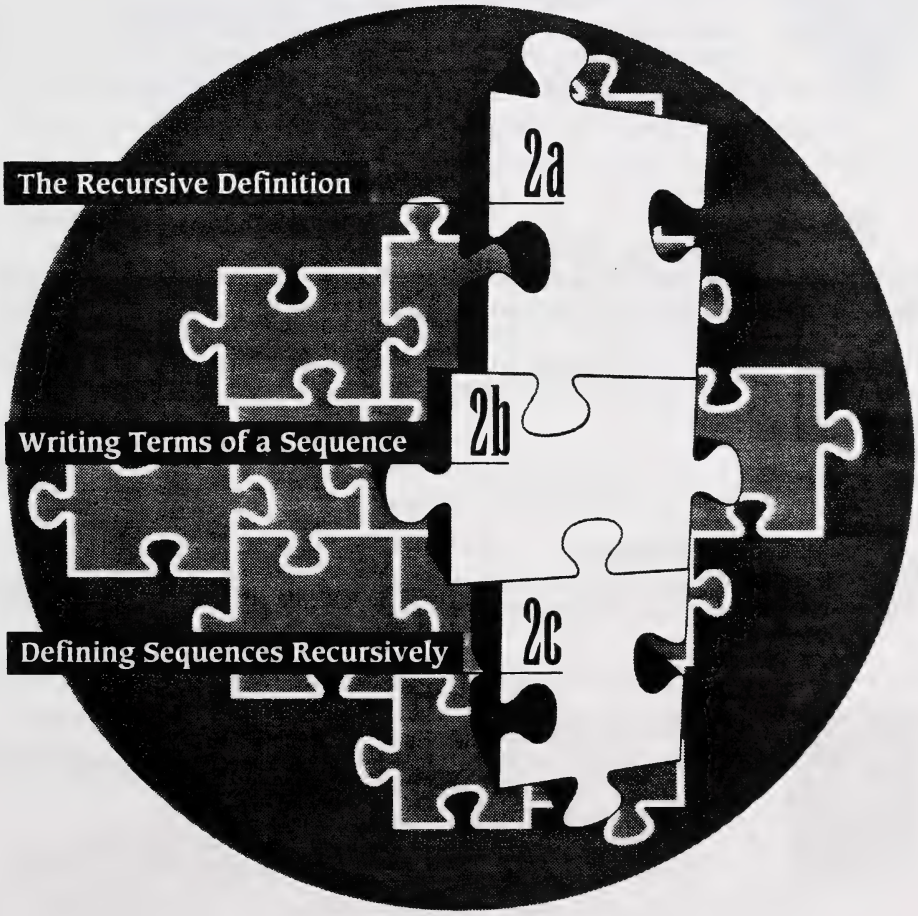
Lesson 2: SEQUENCES: RECURSIVE FORMS

LEARNER EXPECTATIONS

By the end of Lesson 2 you should be able to

- recognize and understand the recursive definition of a sequence
- write the terms of a sequence given its recursive form
- define a sequence recursively given its terms
- give practical applications of the recursive definition of a sequence

This lesson has three parts:



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The Recursive Definition

Watch the Introduction and Video Segment 2a.

Example 1

For the sequence 2, 5, 8, 11, 14, ..., find t_{n-1} and t_{n+2} if $t_n = 11$.

Solution

Method 1

Recall that t_{n-1} means the previous term or the term just prior to t_n. In this case where $t_n = 11$, the previous term is 8. Therefore, $t_{n-1} = 8$.

t_{n+2} means the second term after t_n. If you continue the pattern of this sequence, you obtain 2, 5, 8, 11, 14, 17, ... The second term after $t_n = 11$ is 17. Therefore, $t_{n+2} = 17$.

Method 2

First, observe that 11 is the fourth term; thus, $n = 4$. Next, substitute this value for n into the first desired subscript, $n - 1$, which gives a value of 3. Therefore, $t_{n-1} = t_3 = 8$

Substituting $n = 4$ into $n + 2$, you obtain 6. This means that $t_{n+2} = t_6 = 17$.

Example 2

For the sequence 2, 4, 8, 16, 32, ... one recursive definition is as follows:

$$t_1 = 2,\ t_n = 2t_{n-1},\ n > 1,\ n \in \mathrm{N}$$

Write the definition a different way.

Solution

Instead of using the previous term, t_n, you can use the next term t_{n+1}.

$$t_{n+1} = 2t_n,\ t_1 = 2,\ n \in \mathrm{N}$$

Exercises

1. If $t_n = 10$ in the sequence 2, 6, 10, 14, ... find the following:

   a. t_{n-1}

   b. t_{n-2}

   c. t_{n+3}

2. The sequence 4, −2, 4, −2, 4, ... can be defined by the following:

   $t_1 = 4,\ t_n = 2 - t_{n-1},\ n > 1,\ n \in \mathrm{N}$

   Write a different recursive definition for the sequence.

EXERCISE 2a SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 387 (Question 3)

Writing Terms of a Sequence

Write out the numerical sequence represented by $t_1 = 1, t_n = 5t_{n-1}, n > 1, n \in \mathrm{N}$.

Example 1

Determine the first four terms of the sequence defined recursively by the following:

$t_n = t_{n-1} - 5,\ n > 1,\ n \in \mathrm{N}$, with $t_1 = 3$

Solution

Starting with $n = 2$, the result is as follows:

$$\begin{aligned} t_2 &= t_{2-1} - 5 \\ &= t_1 - 5 \\ &= 3 - 5 \\ &= -2 \end{aligned}$$

For $n = 3$, the result is as follows:

$$\begin{aligned} t_3 &= t_2 - 5 \\ &= -2 - 5 \\ &= -7 \end{aligned}$$

For $n = 4$, the result is as follows:

$$\begin{aligned} t_4 &= t_3 - 5 \\ &= -7 - 5 \\ &= -12 \end{aligned}$$

Thus, the first four terms are $3,\ -2,\ -7$ and -12.

Example 2

Determine the first five terms of the sequence given by $t_{n+2} = t_{n+1} + t_n - 5$, $n \in \mathrm{N}$, when $t_1 = 2$ and $t_2 = 7$. Then give an alternate recursive definition for this sequence using t_n instead of t_{n+2}.

Solution

$t_3 = t_2 + t_1 - 5$
$= 7 + 2 - 5$
$= 4$

$t_4 = t_3 + t_2 - 5$
$= 4 + 7 - 5$
$= 6$

$t_5 = t_4 + t_3 - 5$
$= 6 + 4 - 5$
$= 5$

The first five terms are 2, 7, 4, 6, 5. An alternate definition using t_n is $t_n = t_{n-1} + t_{n-2} - 5$, $t_1 = 2$, $t_2 = 7$, $n > 2$, $n \in N$.

Additional Examples
Mathematics 12: p. 387 (Examples 1, 3)
Mathematics: Principles and Process 12: p. 350 (Example 1), p. 351 (Example 2)
Holtmath 12: p. 419 (Example 2)

Exercises

3. Determine the first five terms for each recursive definition where $n \in$ N.

   a. $t_1 = 3$, $t_n = t_{n-1} + 7$, $n > 1$

   b. $t_1 = 4$, $t_{n+1} = t_n - 6$

   c. $t_1 = -2$, $t_n = t_{n-1} + 8$, $n > 1$

d. $t_1 = -3,\ t_2 = -7,\ t_n = t_{n-1} - t_{n-2},\ n > 2$

4. Complete the following given the recursive definition
$t_1 = 1,\ t_2 = 2,\ t_n = 2t_{n-1} + t_{n-2},\ n > 2,\ n \in N$.

   a. Write out the first five terms.

   b. Express the recursive definition another way using t_{n+2} instead of t_n.

5. Describe both parts of a recursive definition.

EXERCISE 2b SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 387 (Questions 1 and 3)
Mathematics: Principles and Process 12: p. 351 (Questions 1-3), p. 352 (Questions 6, 8, 9)
Holtmath 12: p. 420 (Question 12)

Defining Sequences Recursively

Challenge yourself

Find a recursive definition for the following sequences:

$-2, 4, -8, 16, \ldots$
$-2, 4, 8, 16, \ldots$

MATH FACTOR

Watch Video Segment 2c.

Example 1

Determine a recursive formula for the sequence 2, 6, 18, ...

Solution

Each term is three times the previous term.
Word Statement: Multiply any term by 3 to get the next term.

Recall that t_{n-1} and t_n represent any two consecutive terms of the sequence.

Therefore, a recursive formula is $t_n = 3t_{n-1}$, $n > 1$, $n \in \mathrm{N}$, with $t_1 = 2$.

Example 2

Determine a recursive formula for $3, -1, -5, \ldots$.

Solution

Each term is four less than the previous term.
Word Statement: Subtract 4 from any term to get the next term.

Therefore, a recursive formula is $t_n = t_{n-1} - 4$, $n > 1$, $n \in \mathrm{N}$, with $t_1 = 3$.

TEXT

Additional Examples
Mathematics 12: p. 387 (Example 2)
Mathematics: Principles and Process 12: p. 351 (Example 2b)
Holtmath 12: p. 419 (Example 3)

Exercises

6. Write a word statement and a recursive formula for the following sequences.

   a. 2, 7, 12, ...

   b. 4, −16, 64, ...

   c. 64, 32, 16, 8, ...

   d. 1, 3, 4, 7, 11, 18, ...

   e. 5, 5, 9, 13, 21, 33, ...

7. Tom repaid a loan of $300 in one year by paying monthly payments of $25 as well as monthly interest charges of 1% on the unpaid balance.

   a. Complete the following schedule of payments, unpaid balance and interest charges for the first five months of the loan repayment.

Month	Payment	Interest	Balance
1	$25		$275
2	$25	$2.75	$250
3	$25	$2.50	$225
4			
5			

b. The monthly interest charges show a pattern. Write a word statement to describe this pattern.

c. Write a recursive definition for the amounts of interest paid and describe what n represents in this sequence.

EXERCISE 2c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 387 (Question 2)
Mathematics: Principles and Process 12: p. 352 (Questions 6, 8, 9)
Holtmath 12: p. 420 (Question 13)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. In the next two lessons, you will learn about two particular kinds of sequences: Arithmetic and Geometric. The fundamentals of sequences that you have learned in the first two lessons will help you understand these concepts.

Solutions

EXERCISE 2a

1. a. $t_{n-1} = 6$

   b. $t_{n-2} = 2$

   c. $t_{n+3} = 22$

2. $t_{n+1} = 2 - t_n$, $n \in \mathrm{N}$, $t_1 = 4$

EXERCISE 2b

3. a. 3, 10, 17, 24, 31

   b. 4, −2, −8, −14, −20

   c. −2, 6, 14, 22, 30

   d. −3, −7, −4, 3, 7

4. a. 1, 2, 5, 12, 29

   b. $t_{n+2} = 2t_{n+1} + t_n$, $t_1 = 1$, $t_2 = 2$

5. Both parts of a recursive definition involve the following:

   - the value(s) of the first term(s)
   - a formula to determine the next term based on the previous term(s)

EXERCISE 2c

6. a. $t_n = t_{n-1} + 5$, $t_1 = 2$, $n > 1$, $n \in \mathrm{N}$

   b. $t_n = -4t_{n-1}$, $t_1 = 4$, $n > 1$, $n \in \mathrm{N}$

   c. $t_n = \frac{1}{2}t_{n-1}$, $t_1 = 64$, $n > 1$, $n \in \mathrm{N}$

   d. $t_n = t_{n-1} + t_{n-2}$, $t_1 = 1$, $t_2 = 3$, $n > 2$, $n \in \mathrm{N}$

   e. $t_n = t_{n-1} + t_{n-2} - 1$, $t_1 = 5$, $t_2 = 5$, $n > 2$, $n \in \mathrm{N}$

7. a.

Month	Payment	Interest	Balance
1	$25		$275
2	$25	$2.75	$250
3	$25	$2.50	$225
4	$25	$2.25	$200
5	$25	$2.00	$175

   b. Each term is $0.25 less than the previous term.

   c. $t_n = t_{n-1} - 0.25$, $t_1 = 2.75$, $n > 1$, $n \in N$
   n represents the first, second, third, ... month that interest was paid. For example, $n = 1$ is the first month interest was paid, not the first month that a loan payment of $25.00 was made.

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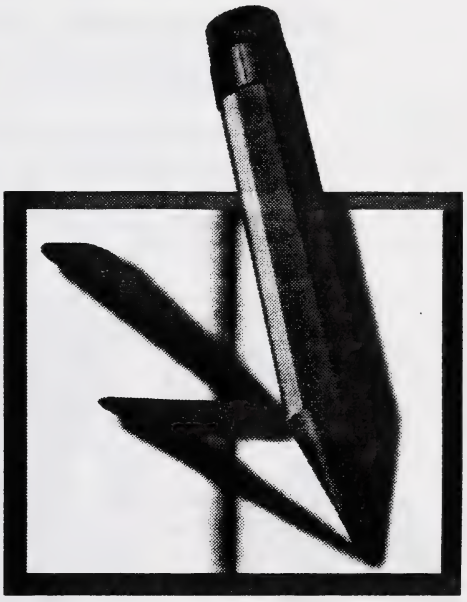
MATH FACTOR

SEQUENCES AND SERIES



Lesson 3:
Arithmetic Sequences

Student Workbook

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This guide accompanies **Arithmetic Sequences** from the Math Factor series.

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Lesson 3: ARITHMETIC SEQUENCES

LEARNER EXPECTATIONS

By the end of Lesson 3 you should be able to

- recognize and understand arithmetic sequences
- apply the general term formula $t_n = a + (n-1)d$ to solve problems

This lesson has four parts:

EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Building Blocks

Do you remember how to solve two equations with two unknowns?

Example

Solve for x and y.

$$3x + 6y = -6$$
$$3x + 2y = -18$$

Solution

Method 1: Subtraction

$$3x + 6y = -6$$
$$3x + 2y = -18$$
$$4y = 12$$
$$y = 3$$

Subtract each term of the bottom equation from the term directly above it.

$$3x + 6(3) = -6$$
$$3x = -24$$
$$x = -8$$

Substitute this value for y into the first equation.

The solution is $x = -8$ and $y = 3$ and can be expressed as the ordered pair $(-8, 3)$.

Method 2: Substitution

$$3x + 2y = -18$$
$$2y = -18 - 3x$$
$$y = -9 - \frac{3}{2}x$$

Solve one of the equations for one of the variables.

$$3x + 6\left(-9 - \frac{3}{2}x\right) = -6$$
$$3x - 54 - 9x = -6$$
$$-6x = -6 + 54$$
$$x = -8$$

Substitute this value in the other equation.

$$\therefore y = -9 - \frac{3}{2}(-8)$$
$$= -9 + 12$$
$$= 3$$

The solution is $(-8, 3)$.

Exercises

1. Solve the following equations for x and y.

   a. $2x + 5y = -4$
      $2x + 6y = 1$

   b. $x - 3y = 6$
      $3x + 3y = -2$

BUILDING BLOCKS SOLUTIONS are at the back of this workbook.

Introduction to Arithmetic Sequences

Find t_{1000} for the arithmetic sequence 10, 12, 14, 16, ...

Watch the Introduction and Video Segment 3a.

Example 1

Determine whether or not the sequence $-10,\ -4,\ 2,\ 8,\ \ldots$ is arithmetic.

Solution

$t_2 - t_1 = -4 - (-10) = 6$
$t_3 - t_2 = -2 - (-4) = 6$
$t_4 - t_3 = 8 - 2 = 6$

Recall that $d = t_n - t_{n-1}$.

By subtracting three pairs of consecutive terms, you obtain a common difference of $d = 6$; therefore, the sequence must be arithmetic.

Example 2

Determine the values of a, d and t_{50} for the arithmetic sequence $2,\ -5,\ -12,\ -19,\ \ldots$.

Solution

Note: Two symbols are commonly used to represent the first term of a sequence: a and t_1.

a is the first term.
$\therefore a = 2$

d is the difference between consecutive terms.

$$\begin{aligned} d &= t_2 - t_1 \\ &= -5 - 2 \\ &= -7 \end{aligned}$$

$$\begin{aligned} t_{50} &= 2 + (50 - 1)(-7) \\ &= 2 + 49(-7) \\ &= -341 \end{aligned}$$

Use $t_n = a + (n-1)d$ to find t_{50}.

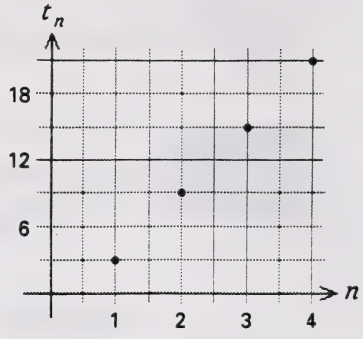
Additional Examples
Mathematics 12: pp. 374, 375
Mathematics: Principles and Process 12: p. 354
Holtmath 12: p. 421 (Example 1)

Exercises

1. The graph on the right shows the first four terms of an arithmetic sequence.

   a. Write out the first four terms.

   b. Determine an expression that defines t_n for this sequence.



2. Determine the value of d, the common difference, for each of the following arithmetic sequences.

   a. 1, 5, 9, ...

   b. 20, 5, −10, ...

   c. $1,\ 1+\sqrt{2},\ 1+2\sqrt{2},\ \ldots$

3. For the arithmetic sequence −6, 2, 10, 18, ..., find the following:

   a. a and d

   b. t_{100}

c. an expression for t_n, the general term

EXERCISE 3a SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 376 (Questions 1-5)
Mathematics: Principles and Process 12: p. 355 (Questions 1, 2)
Holtmath 12: p. 422 (Questions 1, 6, 9)

Using the Formula $t_n = a + (n-1)d$

Challenge yourself

Determine the value of three arithmetic means between \$25 000 and \$38 600.

MATH FACTOR

Watch Video Segment 3b.

Example 1

In the arithmetic sequence 24, 21, 18, ..., $t_n = -39$. Determine the value of n.

Solution

$a = 24$, $d = 21 - 24$, $t_n = -39$ and $n = ?$
$= -3$

Begin by listing the values you do know.

$$t_n = a + (n-1)d$$
$$-39 = 24 + (n-1)(-3)$$
$$-39 = 24 - 3n + 3$$
$$-39 = 27 - 3n$$
$$3n = 66$$
$$n = 22$$

Note: You know the value of t_n but not the number of the term. This means that you are looking for n.

Thus, -39 is the twenty-second term, or t_{22}, in the sequence.

Example 2

Determine the value of four arithmetic means between -8 and 7.

Solution

Think of this as an arithmetic sequence with four missing terms.

-8, ___, ___, ___, ___, 7

$a = -8$, $t_6 = 7$ and $n = 6$

Once you have found the value of d, add it to the first term to find the second term; then add it to the second term to find the third term; and so on.

$$t_n = a + (n-1)d$$
$$t_6 = -8 + (6-1)d$$
$$7 = -8 + (6-1)d$$
$$15 = 5d$$
$$d = 3$$
$$\therefore t_2 = -8 + 3,\ t_3 = -5 + 3,\ t_4 = -2 + 3,\ t_5 = 1 + 3$$
$$= -5 \qquad = -2 \qquad = 1 \qquad = 4$$

The four arithmetic means are -5, -2, 1 and 4.

Example 3

Determine the number of multiples of 5 between 52 and 303.

Solution

The first multiple of 5 in this range is 55.
The second multiple of 5 in this range is 60.
The third multiple of 5 in this range is 65.
The last multiple of 5 in this range is 300.

These numbers form an arithmetic sequence.
55, 60, 65, ..., 300

$a = 55$, $d = 55$ and $t_n = 300$

$$t_n = a + (n-1)d$$
$$300 = 55 + (n-1)(5)$$
$$245 = (n-1)(5)$$
$$49 = n - 1$$
$$n = 50$$

If you can find n for this sequence, you will find the number of multiples of 5.

There are 50 multiples of 5 between 52 and 303.

Example 4

A new car costs \$20 000. During the first year, the value of the car depreciates by \$4000. Each year thereafter, the value of the car decreases by \$1500. After how many years will the car be worth \$5500?

Solution

After one year, the car is worth $\$20\,000 - \$4000 = \$16\,000$.

After two years, the car is worth $\$16\,000 - \$1500 = \$14\,500$.

After three years, the car is worth $\$14\,500 - \$1500 = \$13\,000$.

These values form an arithmetic sequence.
16 000, 14 500, 13 000, ...

$a = 16\,000$, $d = -1500$, $t_n = 5500$ and $n = ?$

$$t_n = a + (n-1)d$$
$$5500 = 16\,000 + (n-1)(-1500)$$
$$5500 = 16\,000 - 1500n + 1500$$
$$-12\,000 = -1500n$$
$$n = 8$$

The value of \$16 000 (after the first year) is used as the first term since the initial value of \$20 000 does not fit the arithmetic pattern.

The car is worth \$5500 after eight years.

Additional Examples
Mathematics 12: p. 375 (Example 2b), p. 376 (Example 4)
Mathematics: Principles and Process 12: p. 354
Holtmath 12: p. 421 (Example 2), p. 422 (Example 3)

Exercises

4. In the arithmetic sequence $-15,\ -9,\ -3,\ \ldots,\ t_n = 63$. Determine the value of n.

5. Determine the number of terms in the arithmetic sequence $8,\ 3,\ -2,\ \ldots,\ -112$.

6. Find the five arithmetic means between 23 and -19.

7. Find the four arithmetic means between $\sqrt{2}$ and $11\sqrt{2}$.

8. Determine the number of multiples of 6 between 40 and 320.

9. A jogger runs 300 m in the first minute, but due to not being in good condition, the jogger runs 20 m less in each succeeding minute. What distance does the jogger run in the ninth minute?

EXERCISE 3b SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 377 (Questions 6-11, 14, 15, 18), p. 378 (Questions 19, 22)
Mathematics: Principles and Process 12: p. 355 (Question 3), p. 356 (Questions 4, 9, 10), p. 358 (Questions 1-8)
Holtmath 12: p. 422 (Question 3), p. 423 (Questions 11-13, 18-21)

Algebraic Sequences

If $x+4, 2x-3, x-5$ are three consecutive terms in an arithmetic sequence, then determine the common difference for the sequence.

Example 1

Find t_n and t_8 for the following arithmetic sequence.

$e-3f, e-6f, e-9f, \ldots$

Solution

$a = e-3f$ and $d = t_2 - t_1$

$= (e-6f)-(e-3f)$

$= -3f$

$t_n = a+(n-1)d$

$= e-3f+(n-1)(-3f)$

$= e-3f-3nf+3f$

$= e-3nf$

$t_8 = a+7d$

$= e-3f+7(-3f)$

$= e-3f-21f$

$= e-24f$

Example 2

If $x+2$, $3x-1$ and $4x-1$ are three consecutive terms in an arithmetic sequence, then find the value of x.

Solution

$d = t_2 - t_1$ and $d = t_3 - t_2$

$= (3x-1)-(x+2)$ $= (4x-1)-(3x-1)$

$= 2x-3$ $= x$

Since d is equal to $2x-3$ and x, it follows that $2x-3$ must be equal to x.

$\therefore 2x-3 = x$

$x = 3$

Check

$3+2, 3(3)-1, 4(3)-1$

$5, 8, 11$

When you substitute 3 for x in the original sequence, the result is an arithmetic sequence.

Exercises

10. If x, $x+y$ and $x+2y$ are the first three terms in an arithmetic sequence, then determine t_n and t_{30}.

11. How many terms are in the following arithmetic sequence?
$5a-3b,\ 4a-2b,\ 3a-b,\ \ldots,\ -5a+7b$

12. If $x+4$, $3x$ and x^2 are three consecutive terms in an arithmetic sequence, then find the following:

    a. the value of x

    b. the value of d

c. the values of the three terms

EXERCISE 3c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 378 (Questions 20, 21)
Mathematics: Principles and Process 12: p. 355 (Question 3), p. 356 (Questions 8, 9)
Holtmath 12: p. 422 (Questions 7, 9k), p. 423 (Questions 12e, 15, 23)

Sequences with Two Unknowns

Find a and d for the arithmetic sequence where the seventh term is 27 and the twelfth term is 52.

Watch Video Segment 3d.

Example

In an arithmetic sequence, $t_8 = -9$ and $t_{17} = -27$. Determine the common difference, d, the first term, a, and the general term, t_n.

Solution

$t_n = a + (n-1)d$
$-9 = a + (8-1)d$
$-9 = a + 7d$ ❶

Substitute $t_8 = -9$ and $n = 8$ into the formula. The resulting equation is labeled ❶.

$t_n = a + (n-1)d$
$-27 = a + (17-1)d$
$-27 = a + 16d$ ❷

Substitute $t_{17} = -27$ and $n = 17$ into the formula. The resulting equation is labeled ❷.

Subtract ❶ from ❷.

$-27 = a + 16d$ ❷
$-9 = a + 7d$ ❶
$-18 = 9d$
$d = -2$

Substitute the value for d in ❶ in order to find a.

$-9 = a + 7(-2)$
$a = 5$

Now substitute the values in the formula in order to find t_n.

$t_n = a + (n-1)d$
$= 5 + (n-1)(-2)$
$= -2n + 7$

Additional Examples
Mathematics 12: p. 376 (Example 3)
Mathematics: Principles and Process 12: p. 355
Holtmath 12: p. 422 (Example 4)

Exercises

13. Determine the values of a, d and t_n for the arithmetic sequence that has the following terms.

    a. $t_5 = -9$ and $t_{20} = 36$

    b. $t_8 = -13$ and $t_{17} = -49$

14. The common difference of an arithmetic sequence is 4. The twenty-first term is two-thirds of the thirty-first term. Find the value of the first term.

EXERCISE 3d SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 377 (Questions 13, 17)
Mathematics: Principles and Process 12: p. 356 (Question 6)
Holtmath 12: p. 423 (Question 14)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. In the next lesson, you will learn about another special kind of sequence – the geometric sequence. Once you have mastered the techniques of working with these special sequences, you will be ready to learn about series in Lessons 5 and 6.

Solutions

BUILDING BLOCKS

1. a. $2x+5y=-4$
$2x+6y=1$
$-y=-5$
$y=5$
$2x+5(5)=-4$
$2x+25=-4$
$2x=-29$
$x=-14\frac{1}{2}$

The solution is $\left(-14\frac{1}{2},-5\right)$.

b. $x-3y=6$
$3x+3y=-2$
$4x=4$
$x=1$
$3(1)+3y=-2$
$3+3y=-2$
$3y=-5$
$y=-\frac{5}{3}$

The solution is $\left(1,-\frac{5}{3}\right)$.

EXERCISE 3a

1. a. From the graph, $t_1=3$, $t_2=9$, $t_3=15$ and $t_4=21$.

b. $t_n=6n-3$

2. a. $d=4$

b. $d=-15$

c. $d=\sqrt{2}$

3. a. $a=-6$ and $d=8$

b. $t_{100}=-6+(100-1)8$
$=786$

c. $t_n=-6+(n-1)(8)$
$=-6+8n-8$
$=8n-14$

EXERCISE 3b

4. $63=-15+(n-1)6$
$63=-15+6n-6$
$84=6n$
$n=14$

63 is the 14th term.

5. $-112=8+(n-1)(-5)$
$-112=8-5n+5$
$-125=-5n$
$n=25$

-112 is the 25th term.
$\therefore$ There are 25 terms in the sequence.

6. 23, ___, ___, ___, ___, ___, -19
$t_7=-19$
$-19=23+(7-1)d$
$-42=6d$
$d=-7$

The five means are 16, 9, 2, -5 and -12.

7. $\sqrt{2}$, ___, ___, ___, ___, $11\sqrt{2}$
$t_6=11\sqrt{2}$
$11\sqrt{2}=\sqrt{2}+(6-1)d$
$10\sqrt{2}=5d$
$d=2\sqrt{2}$

The four means are $3\sqrt{2}$, $5\sqrt{2}$, $7\sqrt{2}$ and $9\sqrt{2}$.

8. The first multiple is 42, the second multiple is 48, the third multiple is 54 and the last multiple is 318.

42, 48, 54, ..., 318

$318 = 42 + (n-1)6$

$276 = (n-1)6$

$46 = n-1$

$n = 47$

There are 47 multiples of 6 between 40 and 320.

9. 300, 280, 260, ..., t_9, ...

$t_9 = 300 + (9-1)(-20)$

$t_9 = 300 + 8(-20) = 140$

The jogger runs 140 m in the ninth minute.

EXERCISE 3c

10. $a = x,\ d = x + y - x$

$= y$

$t_n = a + (n-1)d$

$= x + (n-1)y$

$= x + ny - y$

$\therefore t_{30} = x + 30y - y$

$= x + 29y$

11. $a = 5a - 3b,\ d = (4a - 2b) - (5a - 3b)$

$= -a + b$

$t_n = a + (n-1)d$

$-5a + 7b = 5a - 3b + (n-1)(-a + b)$

$-10a + 10b = (n-1)(-a + b)$

$10(-a + b) = (n-1)(-a + b)$

$10 = n - 1$

$n = 11$

Thus, $-5a + 7b$ is the 11th term so there are 11 terms in this sequence.

12. a. $t_2 - t_1 = t_3 - t_2$

$3x - (x + 4) = x^2 - 3x$

$2x - 4 = x^2 - 3x$

$x^2 - 5x + 4 = 0$

$(x-1)(x-4) = 0$

$x - 1 = 0$ or $x - 4 = 0$

$x = 1$ $\qquad$ $x = 4$

b. $d = -2$ (if $x = 1$) or $d = 4$ (if $x = 4$)

c. 5, 3, 1 (if $x = 1$); 8, 12, 16 (if $x = 4$)

EXERCISE 3d

13. a. $a + 4d = -9$

$a + 19d = 36$

$-15d = -45$

$d = 3$

Find the value of a.

$a + 4(3) = -9$

$a = -9 - 12$

$= -21$

$t_n = -21 + (n-1)3$

$= 3n - 24$

b. $a + 7d = -13$

$a + 16d = -49$

$-9d = 36$

$d = -4$

Find the value of a.

$a + 7(-4) = -13$

$a = -13 + 28$

$= 15$

$t_n = 15 + (n-1)(-4)$

$= -4n + 19$

14. $t_{21} = a + (21-1)(4)$ $\quad t_{31} = a + (31-1)(4)$

$= a + 80$ $\quad = a + 120$

$t_{21} = \frac{2}{3} t_{31}$

$\therefore a + 80 = \frac{2}{3}(a + 120)$

$a + 80 = \frac{2}{3}a + 80$

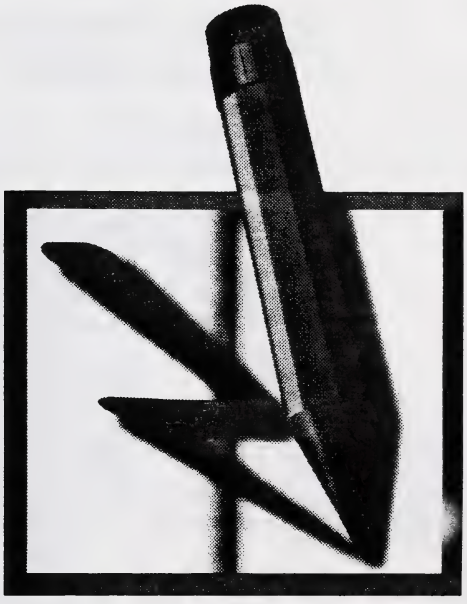
$\frac{1}{3}a = 0$

$a = 0$

2.1997-598 c2 v4

MATH FACTOR

SEQUENCES AND SERIES



Lesson 4:
Geometric Sequences

Student Workbook

Produced by

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for

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This guide accompanies **Geometric Sequences** from the Math Factor series.

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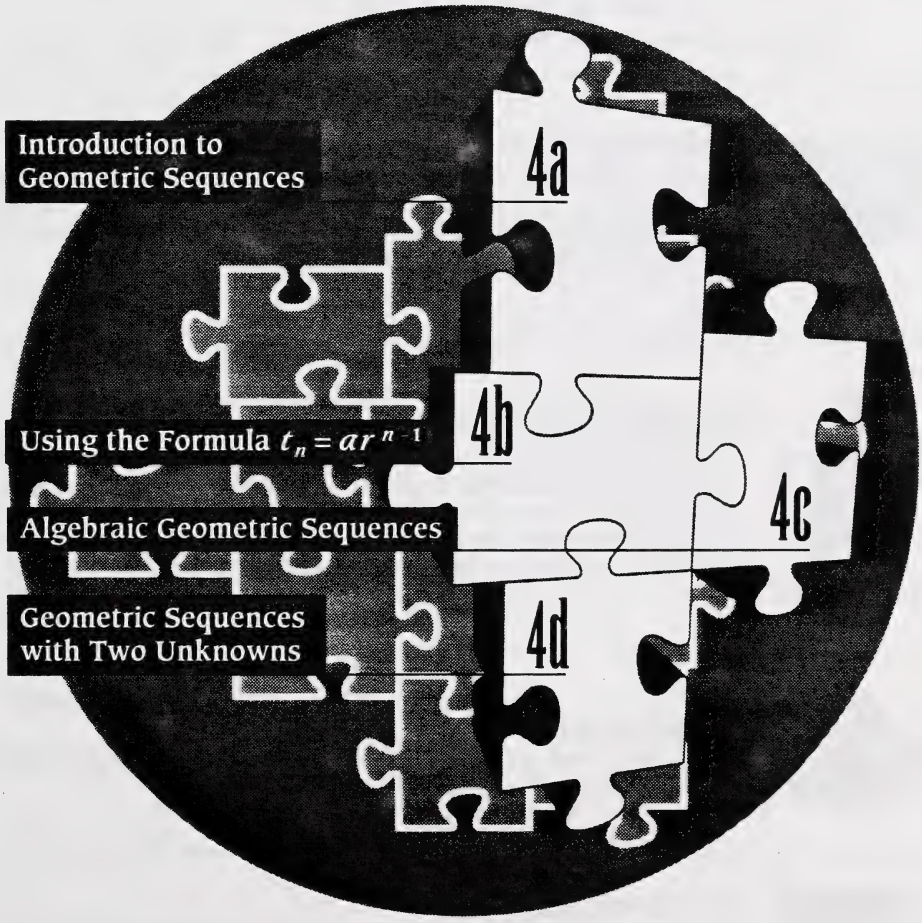
Lesson 4: GEOMETRIC SEQUENCES

LEARNER EXPECTATIONS

By the end of Lesson 4 you should be able to

- recognize and understand geometric sequences
- apply the general term formula $t_n = ar^{n-1}$ to solve problems

This lesson has four parts:



EXPLANATION OF ICONS

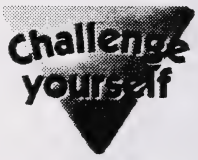
Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught



Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Building Blocks

Do you remember how to solve quadratic equations?

Example

Solve for x in $2x^2 = -5x - 2$.

Solution

$$2x^2 = -5x - 2$$
$$2x^2 + 5x + 2 = 0$$

Begin by moving all terms to one side of the equation.

Method 1: Factoring

$$2x^2 + 5x + 2 = 0$$
$$(x+2)(2x+1) = 0$$
$$x = -2 \text{ or } x = -\frac{1}{2}$$

Method 2: Quadratic Formula

$$2x^2 + 5x + 2 = 0$$
$$a = 2,\ b = 5,\ c = 2$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-5 \pm \sqrt{5^2 - 4(2)(2)}}{2(2)}$$
$$x = \frac{-5 \pm \sqrt{9}}{4}$$
$$x = -2, -\frac{1}{2}$$

Additional Examples
Mathematics 12: p. 27
Mathematics: Principles and Process 12: pp. 48, 49, 53
Holtmath 12: pp. 62, 65

Exercises

1. Solve for x.

   a. $x^2 - 7x + 12 = 0$

   b. $2 = 7x - 6x^2$

BUILDING BLOCKS SOLUTIONS are at the back of this workbook.

Introduction to Geometric Sequences

Watch the Introduction and Video Segment 4a.

Example 1

Determine whether or not the following sequence is geometric.

$$12, -6, 3, -\frac{3}{2}, \ldots$$

Solution

$$\frac{t_2}{t_1} = \frac{-6}{12} = -\frac{1}{2}$$

$$\frac{t_3}{t_2} = \frac{3}{-6} = -\frac{1}{2}$$

$$\frac{t_4}{t_3} = \frac{-\frac{3}{2}}{3} = -\frac{1}{2}$$

Recall that $r = \frac{t_n}{t_{n-1}}$

Observe that by dividing three pairs of consecutive terms, a common ratio of $r = -\frac{1}{2}$ is found. Therefore, the sequence must be geometric.

Example 2

Determine the values of a, r and t_{20} for the following geometric sequence.

$$2, 5, 12.5, \ldots$$

Solution

a is the first term.

$\therefore a = 2$

r is the ratio between two consecutive terms.

$$\therefore \frac{t_2}{t_1} = \frac{5}{2}$$

$$= 2.5$$

Determine t_{20}.

$$t_{20} = 2(2.5)^{20-1}$$
$$= 2(2.5)^{19}$$
$$\doteq 72\,759\,576.14$$

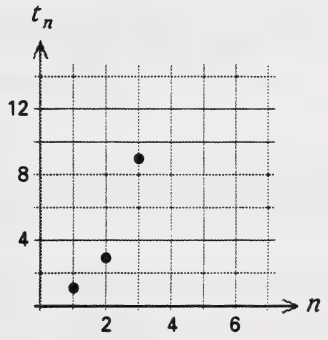
Use $t_n = ar^{n-1}$ to find t_{20}.

TEXT

Additional Examples
Mathematics 12: p. 381, p. 382 (Example 2b)
Mathematics: Principles and Process 12: p. 359
Holtmath 12: p. 427 (Example 1)

Exercises

1. A graph of the first three terms of a geometric sequence is given to the right.



   a. Write down the first three terms.

   b. Write the formula for the general term.

   c. What is the eighth term?

2. Determine whether the following sequences are geometric or non-geometric.

   a. $-1, 3, 9, -27, \ldots$

   b. $\frac{1}{5}, -\frac{2}{5}, \frac{4}{5}, -\frac{8}{5}, \ldots$

3. Determine the common ratio, r, for each of the following geometric sequences.

   a. $4, -\frac{4}{5}, \frac{4}{25}, \ldots$

   b. $1, \sqrt{3}, 3, 3\sqrt{3}, \ldots$

4. For the geometric sequence $4, \frac{4}{5}, \frac{4}{25}, \ldots$, find the following:

   a. a and r

   b. an expression for t_n, the general term

5. Determine the next three terms of the geometric sequence $\frac{1}{3}, -\frac{1}{6}, \frac{1}{12}, \ldots$.

EXERCISE 4a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 383 (Question 1), p. 384 (Questions 2-4, 6), p. 385 (Questions 10, 11a)
Mathematics: Principles and Process 12: p. 360 (Questions 1, 2)
Holtmath 12: p. 428 (Questions 1, 5), p. 429 (Questions 9, 10)

Using the Formula $t_n = ar^{n-1}$

challenge yourself

Find the number of terms in the sequence $\frac{1}{25}, \frac{1}{5}, 1, 5, ..., 312$.

Watch Video Segment 4b.

Example 1

Determine the number of terms in the geometric sequence $-4, 8, -16, ..., 8192$.

This is equivalent to finding which term is 8192.

Solution

$a = -4, r = -2, t_n = 8192$ and $n = ?$

List what is known and the unknown.

$$t_n = ar^{n-1}$$
$$8192 = (-4)(-2)^{n-1}$$

Express -2048 as a power of -2.

$$-2048 = (-2)^{n-1}$$
$$(-2)^{11} = (-2)^{n-1}$$

When the bases are equal, the exponents must also be equal.

$$\therefore 11 = n - 1$$
$$n = 12$$

There are 12 terms in the sequence.

This means that 8192 is the twelfth term in the sequence.

Example 2

Determine the first term of a geometric sequence that has a common ratio of 2 and $t_6 = 160$.

Solution

$a = ?, r = 2, t_n = 160$ and $n = 6$

List what is known and the unknown.

$$t_n = ar^{n-1}$$
$$160 = a(2)^{6-1}$$
$$160 = a(2)^5$$
$$160 = 32a$$
$$a = 5$$

Example 3

Determine the three geometric means between 64 and $\frac{1}{64}$.

Solution

Solve for t_2, t_3 and t_4 in the sequence 64, ___, ___, ___, $\frac{1}{64}$.

$a = 64, t_n = \frac{1}{64}, n = 5$ and $r = ?$

$$t_n = ar^{n-1}$$
$$\frac{1}{64} = 64(r)^{5-1}$$
$$\frac{1}{4096} = r^4$$
$$\pm\frac{1}{8} = r$$

Take the fourth root of both sides of the equation.

If $r = \frac{1}{8}$, then

$t_2 = \frac{1}{8}(64) = 8$ $\quad t_3 = \frac{1}{8}(8) = 1$ $\quad t_4 = \frac{1}{8}(1) = \frac{1}{8}$

In this case, the three geometric means are 8, 1 and $\frac{1}{8}$.

If $r = -\frac{1}{8}$, then

$t_2 = -\frac{1}{8}(64) = -8$ $\quad t_3 = -\frac{1}{8}(-8) = 1$ $\quad t_4 = -\frac{1}{8}(1) = -\frac{1}{8}$

In this case, the three geometric means are -8, 1 and $-\frac{1}{8}$.

Example 4

A certain type of bacteria divides into four every hour. Assuming no bacteria die, what number will be produced from a single bacterium by the eighth hour?

Solution

The sequence is $1, 4, 16, 64, \ldots$.

$a = 1, r = 4, n = 8, t_n = t_8 = ?$

$$t_n = ar^{n-1}$$
$$t_8 = 1(4)^{8-1}$$
$$t_8 = 4^7$$
$$t_8 = 16\,384$$

By the eighth hour, 16 384 bacteria will have been produced.

TEXT

Additional Examples
Mathematics 12: p. 382 (Example 2b), p. 383
Mathematics: Principles and Process 12: pp. 352, 359
Holtmath 12: p. 427 (Example 2), p. 428 (Example 3)

Exercises

6. Determine the number of terms in the geometric sequence $-3, 6, -12, \ldots, 6144$.

7. In the geometric sequence $\frac{1}{66}, -\frac{1}{33}, \frac{2}{33}, \ldots, t_n = -\frac{512}{66}$. Determine the value of n.

8. Determine three geometric means between -5 and -80.

9. Determine four geometric means between 4 and $16\sqrt{2}$. Give your answer as an exact value.

10. A geometric sequence has a common ratio of $-\frac{1}{2}$ and a sixth term of $-\frac{1}{96}$. Determine the first three terms of this sequence.

11. A ball bounces so that each bounce reaches 80% of the height of the previous bounce. If the ball is dropped from a height of 5 m, then correct to the nearest hundredth, how high will it rise after the seventh bounce?

EXERCISE 4b SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 385 (Questions 7-9, 11, 14), p. 386 (Questions 17, 18, 21)
Mathematics: Principles and Process 12: p. 363 (Questions 4, 8-10), p. 361 (Questions 2-6)
Holtmath 12: p. 429 (Questions 11, 12, 15)

Algebraic Geometric Sequences

Challenge yourself

The first three terms of a geometric sequence are $5-\frac{x}{2}$, $x+4$ and $8x+2$. Find the value of x.

Watch Video Segment 4c.

Example 1

For the geometric sequence $x^2, 2x^3y, 4x^4y^2, \ldots$, find t_n and t_{15}.

Solution

$$a = x^2, r = \frac{t_2}{t_1}$$
$$= \frac{2x^3y}{x^2}$$
$$= 2xy$$

$$t_n = ar^{n-1}$$
$$= (x^2)(2xy)^{n-1}$$
$$= (x^2)(2^{n-1})(x^{n-1})(y^{n-1})$$
$$= 2^{n-1}x^{n+1}y^{n-1}$$

$$t_{15} = 2^{15-1}x^{15+1}y^{15-1}$$
$$= 2^{14}x^{16}y^{14}$$

EXCELLENCE

Example 2

If $(2x+4)$, $(x-10)$ and $(x-1)$ are three consecutive terms in a geometric sequence, then determine the value or values of x.

Solution

$$\frac{x-10}{2x+4} = \frac{x-1}{x-10}$$

Recall that $r = \frac{t_2}{t_1} = \frac{t_3}{t_2}$.

$$(x-10)(x-10) = (x-1)(2x+4)$$

Multiply each side by $(x-10)(2x+4)$.

$$x^2 - 20x + 100 = 2x^2 + 2x - 4$$
$$x^2 + 22x - 104 = 0$$
$$(x-4)(x+26) = 0$$

$x = 4$ or $x = -26$

These solutions can be checked by substituting into the original sequence.

Exercises

12. Show that each of the solutions for Example 2 does form a geometric sequence.

13. Find t_n and t_{10} for the geometric sequence $x, bx^3, b^2x^5, \ldots$. Give your answer in terms of x and b.

14. Given that $(x+1), (x+7)$ and $(2x+14)$ are three consecutive terms in a geometric sequence, find the following:

    a. the value or values of x

    b. the values of the three terms and the common ratio of the sequence

EXERCISE 4c SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 386 (Questions 19, 20)
Mathematics: Principles and Process 12: p. 361 (Questions 3, 7)
Holtmath 12: p. 429 (Questions 10f, 14)

Geometric Sequences with Two Unknowns

Challenge yourself

Find t_n for the geometric sequence where the second term is 24 and the sixth term is 1944.

MATH FACTOR

Watch Video Segment 4d.

Example 1

If $x, 2, y$ and $4y$ are four consecutive terms in a geometric sequence, then find the values of x and y.

Solution

If $x, 2, y$ and $4y$ form a geometric sequence, then

$$\frac{y}{2} = \frac{4y}{y}$$
$$y^2 = 8y$$
$$y^2 - 8y = 0$$
$$y(y-8) = 0$$

$y = 0$ or $y = 8$

If $y = 8$, then $r = 4$. It follows that $x = \frac{t_2}{r} = \frac{1}{2}$.

If $y = 0$, then there can be no solution for x; thus, $y = 0$ is extraneous.

Therefore, $x = \frac{1}{2}$ and $y = 8$, and the resulting sequence is $\frac{1}{2}, 2, 8, 32$.

Example 2

In a geometric sequence, the sixth term is 972 and the eighth term is 8748. Determine the value or values of the first term, the value or values of the common ratio and an expression or expressions for t_n.

Solution

$$t_n = ar^{n-1} \qquad t_n = ar^{n-1}$$
$$t_6 = ar^{6-1} \qquad t_8 = ar^{8-1}$$
$$972 = ar^5 \qquad 8748 = ar^7$$

$$\frac{ar^7}{ar^5} = \frac{8748}{972}$$
$$r^2 = 9$$
$$r = \pm 3$$

If $r = 3$ and $t_6 = 972$, then $972 = a(3)^5$ and $t_n = 4(3)^{n-1}$.

$$972 = 243a$$
$$a = 4$$

If $r = -3$ and $t_6 = 972$, then $972 = a(-3)^5$ and $t_n = -4(-3)^{n-1}$.

$$972 = -243a$$
$$a = -4$$

Additional Examples
Mathematics 12: p. 382 (Question 3)
Mathematics: Principles and Process 12: p. 360
Holtmath 12: p. 428 (Question 4)

Exercises

15. If $2x, 2y, xy$ and $5y$ are four consecutive terms in a geometric sequence, then find the values of x and y.

EXCELLENCE

16. Determine the values of the first three terms for geometric sequences that have the following terms.

a. $t_3 = 1, t_6 = -64$

b. $t_6 = 80, t_{10} = 1280$

EXERCISE 4d SOLUTIONS are at the back of this workbook.

TEXT

Additional Examples
Mathematics 12: p. 385 (Questions 15, 16), p. 386 (Questions 24-26, 28, 29)
Mathematics: Principles and Process 12: p. 361 (Questions 5, 6)
Holtmath 12: p. 429 (Question 13)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. In the next lessons, you will learn about arithmetic and geometric series. A series is formed when the terms of a sequence are added.

Solutions

BUILDING BLOCKS

1. a. $x^2 - 7x + 12 = 0$
$(x-3)(x-4) = 0$
$x = 3$ or $x = 4$

b. $2 = 7x - 6x^2$
$6x^2 - 7x + 2 = 0$
$(2x-1)(3x-2) = 0$
$x = \frac{1}{2}$ or $x = \frac{2}{3}$

EXERCISE 4a

1. a. 1, 3, 9

b. $t_n = 3^{n-1}$

c. $t_8 = 3^{8-1}$
$= 3^7$
$= 2187$

2. a. non-geometric

b. geometric

3. a. $r = -\frac{1}{5}$

b. $r = \sqrt{3}$

4. a. $a = 4, r = \frac{\frac{4}{5}}{4}$
$= \frac{1}{5}$

b. $t_n = 4\left(\frac{1}{5}\right)^{n-1}$

5. $a = \frac{1}{3}, r = \frac{-\frac{1}{6}}{\frac{1}{3}}$
$= -\frac{1}{2}$

$t_4 = -\frac{1}{2}t_3$
$= -\frac{1}{2}\left(\frac{1}{12}\right)$
$= -\frac{1}{24}$

$t_5 = -\frac{1}{2}t_4$
$= -\frac{1}{2}\left(-\frac{1}{24}\right)$
$= \frac{1}{48}$

$t_6 = -\frac{1}{2}t_5$
$= -\frac{1}{2}\left(\frac{1}{48}\right)$
$= -\frac{1}{96}$

EXERCISE 4b

6. $a = -3, t_n = 6144, r = -2$
$6144 = -3(-2)^{n-1}$
$-2048 = (-2)^{n-1}$
$(-2)^{11} = (-2)^{n-1}$
$\therefore n - 1 = 11$
$n = 12$
There are 12 terms.

7. $a = \frac{1}{66}, r = -2, t_n = -\frac{512}{66}$
$-\frac{512}{66} = \frac{1}{66}(-2)^{n-1}$
$-512 = (-2)^{n-1}$
$(-2)^9 = (-2)^{n-1}$
$\therefore n - 1 = 9$
$n = 10$
$-\frac{512}{66}$ is the tenth term.

8. $-5, \_, \_, \_, -80$

$t_5 = -80$

$-80 = -5(r)^{5-1}$

$16 = r^4$

$r = \pm 2$

If $r = 2$, then the means are $-10, -20$ and -40.

If $r = -2$, then the means are $10, -20$ and 40.

9. $4, \_, \_, \_, \_, 16\sqrt{2}$

$t_6 = 16\sqrt{2}$

$16\sqrt{2} = 4(r)^{6-1}$

$4\sqrt{2} = r^5$

$r = \sqrt{2}$

The means are $4\sqrt{2}$, 8, $8\sqrt{2}$ and 16.

10. $-\frac{1}{96} = a\left(-\frac{1}{2}\right)^{6-1}$

$-\frac{1}{96} = -\frac{1}{32}a$

$a = \frac{1}{3}$

$t_2 = \frac{1}{3}\left(-\frac{1}{2}\right) = -\frac{1}{6}$ $\qquad$ $t_3 = -\frac{1}{6}\left(-\frac{1}{2}\right) = \frac{1}{12}$

The first three terms are $\frac{1}{3}$, $-\frac{1}{6}$ and $\frac{1}{12}$.

11. $t_1 = 5$ m, $r = 0.80$

The height reached after the seventh bounce is actually t_8 in the sequence.

$t_8 = 5(0.80)^7$

$= 1.05$ m

Thus, the ball will reach a height of 1.05 m.

EXERCISE 4c

12. If $x = 4$, then $2(4)+4, 4-10,$ $(4-1)$ or $12, -6, 3$, which is geometric with $r = -\frac{1}{2}$.

If $x = -26$, then $2(-26)+4,$ $-26-10, -26-1$ or $-48, -36, -27,$ which is geometric with $r = -\frac{3}{4}$.

13. $a = x$ and $r = \frac{bx^3}{x}$

$= bx^2$

$t_n = ar^{n-1}$

$= x\left(bx^2\right)^{n-1}$

$= xb^{n-1}x^{2n-2}$

$= b^{n-1}x^{2n-1}$

$t_{10} = b^{10-1}x^{2(10)-1}$

$t_{10} = b^9x^{19}$

14. a. $\frac{t_2}{t_1} = \frac{t_3}{t_2}$

$\frac{x+7}{x+1} = \frac{2x+14}{x+7}$

$x^2 + 14x + 49 = 2x^2 + 16x + 14$

$x^2 + 2x - 35 = 0$

$(x+7)(x-5) = 0$

$x = -7$ or $x = 5$

b. If $x = -7$, then

$-7+1, -7+7, 2(-7)+14$

or $-6, 0, 0$.

This is an extraneous solution as the terms do not form a geometric sequence.

If $x = 5$, then

$5+1, 5+7, 2(5)+14$

or $6, 12, 24$.

The common ratio is 2.

EXERCISE 4d

15. $\frac{2y}{2x} = \frac{xy}{2y}$ $\qquad$ $\frac{xy}{2y} = \frac{5y}{xy}$

$4y^2 = 2x^2y$ $\qquad$ $x^2y^2 = 10y^2$

$4y = 2x^2$ $\qquad$ $x^2 = 10$

$4y = 2\left(\sqrt{10}\right)^2$ $\qquad$ $x = \pm\sqrt{10}$

$4y = 20$

$y = 5$

When $x = +\sqrt{10}$, the terms are:

$2\sqrt{10}, 10, 5\sqrt{10}, 25$

or

When $x = -\sqrt{10}$, the terms are:

$-2\sqrt{10}, 10, -5\sqrt{10}, 25$

The sequence is geometric with $r = \pm\frac{\sqrt{10}}{2}$.

16. a. $\frac{ar^5}{ar^2} = \frac{-64}{1}$

$r^3 = -64$

$r = -4$

Find the first term.

$t_n = ar^{n-1}$

$t_3 = a(-4)^{3-1}$

$1 = a(-4)^2$

$a = \frac{1}{16}$

Therefore, the first three terms are $\frac{1}{16}$, $-\frac{1}{4}$ and 1.

b. $\frac{ar^9}{ar^5} = \frac{1280}{80}$

$r^4 = 16$

$r = \pm 2$

If $r = 2$, then $80 = a(2)^5$ and $a = \frac{5}{2}$.

$a = \frac{5}{2}$; therefore, the first three terms are $\frac{5}{2}, 5, 10$.

If $r = -2$, then $a = -\frac{5}{2}$ and the first three terms are $-\frac{5}{2}, 5, -10$.

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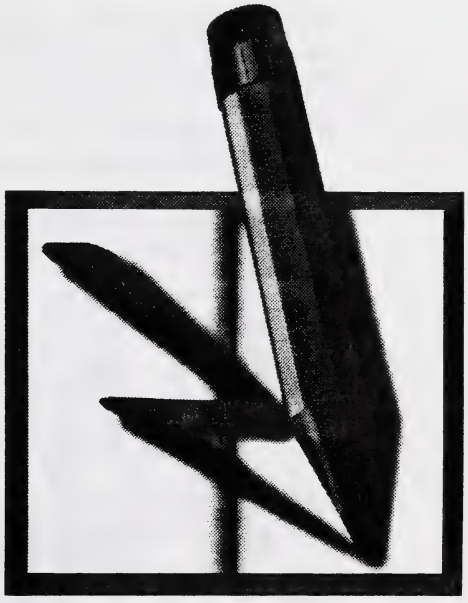
MATH FACTOR

SEQUENCES AND SERIES



Lesson 5:
Arithmetic Series

Student Workbook

Produced by

THE EDUCATION STATION

ACCESS

for

Alberta
EDUCATION

This guide accompanies **Arithmetic Series** from the Math Factor series.

Writer: Jim Ashton
Editor: Wilma Gonsalves
Producer: Katherine Rankin

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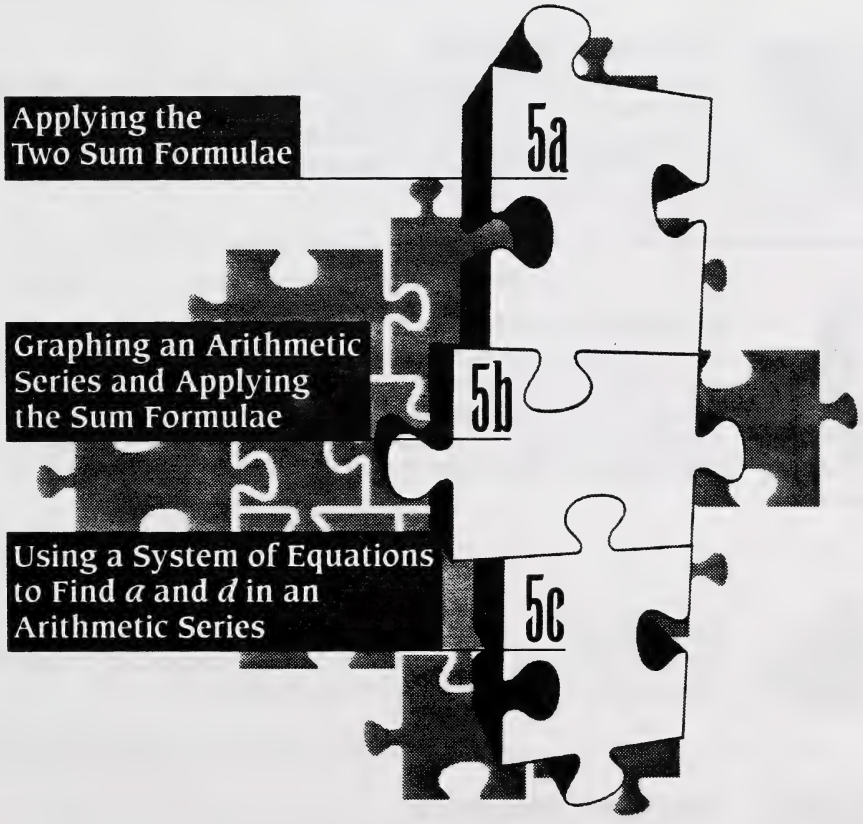
Lesson 5: ARITHMETIC SERIES

LEARNER EXPECTATIONS

By the end of Lesson 5 you should be able to

- demonstrate an understanding that an arithmetic series is the sum of the terms of an arithmetic sequence
- apply both forms of the sum formulae to a variety of problem-solving situations

This lesson has three parts:


Applying the Two Sum Formulae
5a
Graphing an Arithmetic Series and Applying the Sum Formulae
5b
Using a System of Equations to Find a and d in an Arithmetic Series
5c

EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
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Textbook Reference:

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- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Building Blocks

Do you remember how to solve a system of equations (two equations with two unknowns)? The method shown in the example is the comparison method. Other methods include substitution and addition or subtraction.

Example

Solve for x and y in the following system of equations:

$$-2x + y = 10$$
$$x + y = 19$$

Solution

Equate both equations to y.

$$y = 2x + 10$$
$$y = -x + 19$$

Solve for x.

$$2x + 10 = -x + 19$$
$$3x = 9$$
$$x = 3$$

The left sides of these equations are both equal to y; thus, the two right sides must be equal.

Substitute $x = 3$ into either equation and solve for y.

$$x + y = 19$$
$$3 + y = 19$$
$$y = 16$$

Check your answer by substituting your answers for x and y into both original equations.

L.S.	R.S.
$-2x + y$	10
$-2(3) + 16$	

L.S. = R.S.

L.S.	R.S.
$x + y$	19
$3 + 16$	
19	

L.S. = R.S.

$\therefore$ The solution is $(3, 16)$.

Exercises

1. Solve the following systems of equations.

   a. $2x + y = 4$
      $3x + y = 8$

   b. $6a + 3d = 18$
      $a + d = -12$

   c. $2m - n = 12$
      $m + n = 21$

   d. $2a - n = 200$
      $a + n = 103$

BUILDING BLOCKS SOLUTIONS are at the back of this workbook.

Applying the Two Sum Formulae

Challenge yourself

Determine the sum of the first twelve terms of the series $-5+(-15)+(-25)+....$

MATH FACTOR

Watch the Introduction and Video Segment 5a.

Example 1

Determine the sum of the first 15 terms, S_{15}, of the arithmetic series $25+22+19+....$

Solution

$a=25$, $d=-3$ and $n=15$

List the information that you know; then choose a formula that uses this information.

$S_n = \dfrac{n(a+t_n)}{2}$ needs n, a and t_n.

$S_n = \dfrac{n[2a+(n-1)d]}{2}$ needs n, a and d.

$$S_n = \frac{n[2a+(n-1)d]}{2}$$

$$S_{15} = \frac{15[2(25)+(15-1)(-3)]}{2}$$

$$= \frac{15(50-42)}{2}$$

$$= 60$$

Substitute and solve.

Example 2

Find the sum of 100 arithmetic means between 1 and 600.

Solution

Find the sum of the entire series, including 1 and 600; then subtract 601 to find the sum of the means. $1+...+600$ has 102 terms.

$a = 1$, $n = 102$ and $t_n = 600$

$$S_n = \frac{n(a + t_n)}{2}$$

$$S_{102} = \frac{102(1 + 600)}{2}$$

$$= 30\,651$$

Substitute and solve.

The sum of the means is $30\,651 - 601 = 30\,050$.

Additional Examples
Mathematics 12: p. 394 (Example 1), p. 395 (Examples 2, 3)
Mathematics: Principles and Process 12: p. 368 (Example 1)
Holtmath 12: p. 424 (Example 1), p. 425 (Example 2)

Exercises

1. Find the sum of the first 25 terms of the arithmetic series $8 + 11 + 14 + \ldots$.

2. Find the sum of the arithmetic series $1 + 5 + 9 + \ldots + 85$.

3. A pile of logs at a sawmill has 100 logs in the bottom row, 99 in the next row and so on. If there are 40 logs in the top row, how many logs are there in the entire pile?

4. A local golf pro decided to work on his putting during the off-season, so he set up a practice putting green in his basement. He began by making one putt on the first day, and each day after that he increased the number of putts by two.

   a. At the end of 90 days, what is the total number of putts made?

   b. How many putts does he have to make before quitting practice on the 90th day?

5. A grandfather clock chimes once for one o'clock, twice for two o'clock, three times for three o'clock and so on. In addition, it chimes once each quarter hour, twice each half hour and three times each three-quarter hour. How many times does the clock chime in a day?

6. If $t_n = -4n + 7$ for some arithmetic series, then determine an expression for S_n.

7. If $S_n = 2n^2 + 3n$ is an expression that describes the sum of n terms of an arithmetic series, then determine an expression for t_n of this series.

EXERCISE 5a SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 395 (Questions 1, 3-5, 9, 17, 18)
Mathematics: Principles and Process 12: p. 368 (Questions 1-5), p. 371 (Questions 1-4)
Holtmath 12: p. 425 (Questions 6, 7, 11)

Graphing an Arithmetic Series and Applying the Sum Formulae

If $S_n = -99$ for the arithmetic series 16 + 11 + 6 + ..., then find the value of n.

Watch Video Segment 5b.

Example 1

If $S_n = 828$ for the arithmetic series 3 + 6 + 9 + 12 + ..., then determine the value of n.

Solution

$a = 3,\ d = 3,\ S_n = 828$ and $n = ?$ — List the information that you know.

$$S_n = \frac{n[2a + (n-1)d]}{2}$$

$$828 = \frac{n[2(3) + (n-1)(3)]}{2}$$ — Substitute and solve.

$$1656 = n(6 + 3n - 3)$$

$$1656 = n(3n + 3)$$

$$1656 = 3n^2 + 3n$$

$3n^2 + 3n - 1656 = 0$ — Divide both sides by 3.

$n^2 + n - 552 = 0$

$(n + 24)(n - 23) = 0$ — Factor (or use the quadratic formula) to solve.

$n = -24$ or $n = 23$

Since you can't have a negative number of terms, the only valid answer is $n = 23$.

Example 2

What is the sum of the multiples of 5 between 51 and 202?

Solution

$a = 55,\ d = 5,\ t_n = 200,\ n = ?$ and $S_n = ?$

List the information that you know.

$t_n = a + (n-1)d$

$200 = 55 + (n-1)5$

You need to calculate the number of terms, n, in order to calculate S_n.

$150 = 5n$

$30 = n$

$$S_n = \frac{n(a + t_n)}{2}$$

$$= \frac{30(55 + 200)}{2}$$

$$= 3825$$

Now that n is known, choose a formula and solve.

Therefore the sum of the multiples of 5 between 51 and 202 is 3825.

TEXT

Additional Examples
Mathematics: Principles and Process 12: p. 368 (Example 2)
Holtmath 12: p. 425 (Example 3)

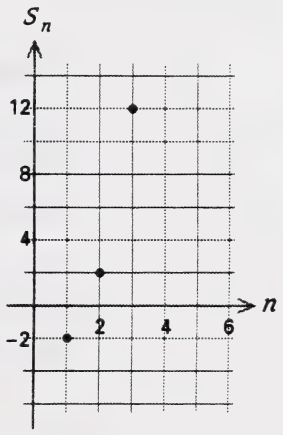
Exercises

8. If $S_n = 498$ for the arithmetic series 14 + 19 + 24 + ..., then determine the number of terms in the sequence.

9. What is the sum of all the multiples of 4 between 45 and 601?

10. A new movie theatre has 20 rows of seats. Each row has four more seats than the previous row. If there are a total of 1160 seats in the theatre, how many seats are in the first row?

11. The graph shows the sum of n terms in an arithmetic series as a function of n, $n \in$ N. Each point on the graph is of the form (n, S_n), where S_n is the sum of n terms in the series. For example, (2, 2) means $S_2 = 2$. Write out the first four terms in the series and find the general formula for S_n.



EXERCISE 5b SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 396 (Questions 13, 15, 16)
Mathematics: Principles and Process 12: p. 369 (Question 10a)
Holtmath 12: p. 426 (Question 14)

Using a System of Equations to Find a and d in an Arithmetic Series

Challenge yourself

In an arithmetic series, the sum of the first four terms is 426, and the sum of the first eight terms is 804. Find the common difference, S_{13}, and the first five terms.

Example

The sum of the first three terms of an arithmetic series is 45. The sum of the first seven terms of the same series is 203. Find the first three terms of the series.

Solution

To create the first three terms of the series, you need to know a and d.

The formula $S_n = \dfrac{n[2a+(n-1)d]}{2}$ uses a and d.

$$S_n = \frac{n[2a+(n-1)d]}{2} \qquad S_n = \frac{n[2a+(n-1)d]}{2}$$

$$45 = \frac{3[2a+(3-1)d]}{2} \qquad 203 = \frac{7[2a+(7-1)d]}{2}$$

$$90 = 6a + 6d \qquad 406 = 14a + 42$$

$$-d + 15 = a \qquad -3d + 29 = a$$

Solve for d.

$$-d + 15 = -3d + 29$$
$$2d = 14$$
$$d = 7$$

Recall systems of equations. You may wish to use one of the other methods to solve systems.

Solve for a.

$$-d + 15 = a$$
$$-7 + 15 = a$$
$$8 = a$$

The first three terms of the series are 8 + 15 + 22.

Exercises

12. The sum of the first five terms of an arithmetic series is 60. The sum of the first nine terms of the same series is 72. Find the first four terms of the series.

13. The sum of the first 14 terms of an arithmetic series is -28. The sum of the first 26 terms of the same series is 572. Find the value of the first term, the value of the common difference and S_{10} for this series.

14. In an arithmetic series, the sum of $t_3 + t_4 + t_5 = 24$ and the sum of $t_8 + t_9 + t_{10} = 54$. Find the first three terms of the series.

EXERCISE 5c SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 396 (Questions 11, 12, 14)
Holtmath 12: p. 426 (Questions 18, 20)
Mathematics: Principles and Process 12: p. 369 (Question 8)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. In the next lesson, you will learn about geometric series, which can be applied to problems in mathematics, business and the sciences.

Solutions

BUILDING BLOCKS

1. a. $y = -2x + 4$ $\quad y = -3x + 8$

$-2x + 4 = -3x + 8$

$x = 4$

$2(4) + y = 4$

$y = -4$

Check

L.S.	R.S.
$2(4) - 4$	4

L.S. = R.S.

L.S.	R.S.
$3(4) - 4$	8

L.S. = R.S.

$\therefore$ The solution is $(4, -4)$

b. $3d = -6a + 18$ $\quad d = -a - 12$

$d = -2a + 6$

$-2a + 6 = -a - 12$

$a = 18$

$18 + d = -12$

$d = -30$

Check

L.S.	R.S.
$6(18) + 3(-30)$	18

L.S. = R.S.

L.S.	R.S.
$18 - 30$	-12

L.S. = R.S.

$\therefore$ The solution is $(18, -30)$

c. $n = 2m - 12$ $\quad n = -m + 21$

$2m - 12 = -m + 21$

$3m = 33$

$m = 11$

$11 + n = 21$

$n = 10$

Check

L.S.	R.S.
$2(11) - 10$	12

L.S. = R.S.

L.S.	R.S.
$11 + 10$	21

L.S. = R.S.

$\therefore$ The solution is $(11, 10)$

d. $n = 2a - 200$ $\quad n = -a + 103$

$2a - 200 = -a + 103$

$a = 101$

$101 + n = 103$

$n = 2$

Check

L.S.	R.S.
$2(101) - 2$	200

L.S. = R.S.

L.S.	R.S.
$11 + 10$	21

L.S. = R.S.

$\therefore$ The solution is $(101, 2)$

EXERCISE 5a

1. $a = 8$, $d = 3$ and $n = 25$

$$S_n = \frac{n\left[2a + (n-1)d\right]}{2}$$

$$S_{25} = \frac{25\left[2(8) + (25-1)3\right]}{2}$$

$$= 1100$$

2. $a = 1$, $d = 4$ and $t_n = 85$

$t_n = a + (n-1)d$

$85 = 1 + (n-1)(4)$

$n = 22$

$$S_n = \frac{n(a + t_n)}{2}$$

$$S_{22} = \frac{22(1 + 85)}{2}$$

$$= 946$$

3. $a = 100$, $d = -1$ and $t_n = 40$

$t_n = a + (n-1)d$

$40 = 100 + (n-1)(-1)$

$n = 61$

$$S_{61} = \frac{61(100 + 40)}{2}$$

$$= 4270$$

4. a. $a = 1,\ d = 2$ and $n = 90$

$$S_{90} = \frac{90\left[2(1)+(90-1)2\right]}{2}$$
$$= 8100$$

b. $t_n = a+(n-1)d$

$$t_{90} = 1+(90-1)(2)$$
$$= 179$$

5. In the first 12 hours (a.m.), it chimes $7+8+9+\ldots+18$ times.

$a = 7,\ d = 1$ and $n = 12$

$$S_{12} = \frac{12\left[2(7)+(12-1)1\right]}{2}$$
$$= 150$$

The second 12 hours (p.m.) are a repeat of the first 12. Therefore, $S_{24} = 300$.

6. $t_n = -4n+7,\ t_1 = 3,\ t_2 = -1$ and $t_3 = -5$

$$S_n = \frac{n\left[3+(-4n)+7\right]}{2}$$
$$= \frac{n(-4n+10)}{2}$$
$$= -2n^2+5n$$

7. $S_1 = 2(1)^2+3(1) = 5 \quad \therefore t_1 = 5$

$S_2 = 2(2)^2+3(2) = 14 \quad \therefore t_2 = 14-5 = 9$

$S_3 = 27 \quad \therefore t_3 = 27-14 = 13$

Thus, the series is $5+9+13+\ldots$ (arithmetic).

$t_n = a+(n-1)d$

$= 5+(n-1)4$

$= 4n+1$

EXERCISE 5b

8. $a = 14,\ d = 5,\ S_n = 498$ and $n = ?$

$$S_n = \frac{n\left[2a+(n-1)d\right]}{2}$$
$$498 = \frac{n\left[28+(n-1)5\right]}{2}$$
$$996 = 5n^2+23n$$
$$5n^2+23n-996 = 0$$
$$(5n+83)(n-12) = 0$$

$n = 12$ (since n cannot be negative)

9. $48+52+56+\ldots+600$

$a = 48,\ d = 4,\ S_n = ?,\ t_n = 600$

$$600 = 48+(n-1)(4)$$
$$4n+44 = 600$$
$$n = 139$$
$$S_{139} = \frac{139(48+600)}{2}$$
$$= 45\,036$$

10. $n = 20,\ S_n = 1160,\ d = 4,\ a = ?$

$$1160 = \frac{20\left[2a+(19)(4)\right]}{2}$$
$$a = 20$$

11. $S_1 = -2 \quad \therefore t_1 = -2$

$S_2 = 2 \quad \therefore t_2 = 4$

$S_3 = 12 \quad \therefore t_3 = 12-2 = 10$

The first four terms of the series are $-2+4+10+16$.

$$S_n = \frac{n\left[-4+(n-1)6\right]}{2}$$
$$= \frac{n\left[6n-10\right]}{2}$$
$$= n(3n-5)$$

EXERCISE 5c

12. $S_5 = 60$ $\qquad$ $S_9 = 72$

$$60 = \frac{5(2a+4d)}{2} \qquad 72 = \frac{9(2a+8d)}{2}$$

$120 = 10a + 20d$ ❶ $\qquad$ $144 = 18a + 72d$ ❷

$12 = a + 2d$ (Divide ❶ by 10.)

$8 = a + 4d$ (Divide ❷ by 18.)

Solve the system of two equations to get $a = 16$ and $d = -2$.

The first four terms of the series are 16 + 14 + 12 + 10.

13. $S_{14} = -28$ and $S_{26} = 572$

$$-28 = \frac{14(2a+13d)}{2} \qquad 572 = \frac{26(2a+25d)}{2}$$

$-28 = 7(2a+13d)$ $\qquad$ $572 = 13(2a+25d)$

$-4 = 2a + 13d$ ❶ $\qquad$ $44 = 2a + 25d$ ❷

Subtract ❶ from ❷.

$$\begin{array}{r} 44 = 2a + 25d \\ -4 = 2a + 13d \\ \hline 48 = 12d \end{array}$$

$d = 4$

Solve for a by substituting $d = 4$ into ❶.

$-4 = 2a + 13(4)$

$a = -28$

$$S_{10} = \frac{10[-56 + 9(4)]}{2}$$

$= -100$

14. $t_n = a + (n-1)d$

$t_3 + t_4 + t_5 = 24$ $\qquad$ $t_8 + t_9 + t_{10} = 54$

$a + 2d + a + 3d + a + 4d = 24$ $\qquad$ $a + 7d + a + 8d + a + 9d = 54$

$3a + 9d = 24$ ❶ $\qquad$ $3a + 24d = 54$ ❷

$3a + 9d = 24$

$3a + 24d = 54$

Solve the system of two equations to get $d = 2$ and $a = 2$.

The first three terms are 2 + 4 + 6.

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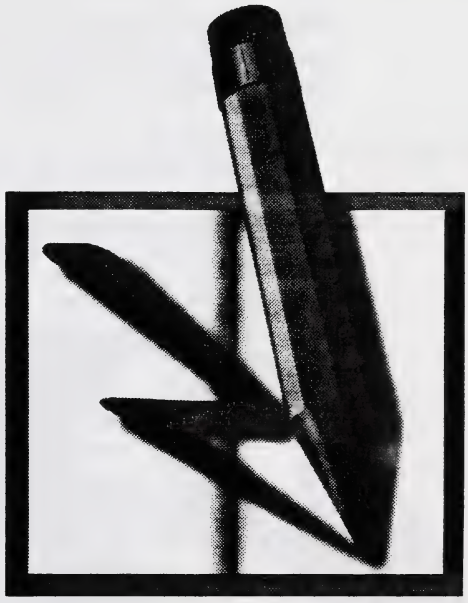
SEQUENCES AND SERIES



Lesson 6:
Geometric Series

Student Workbook

Produced by

THE EDUCATION STATION

for

Alberta EDUCATION

This guide accompanies **Geometric Series** from the Math Factor series.

Writer: Chris Ream
Editor: Wilma Gonsalves
Producer: Katherine Rankin

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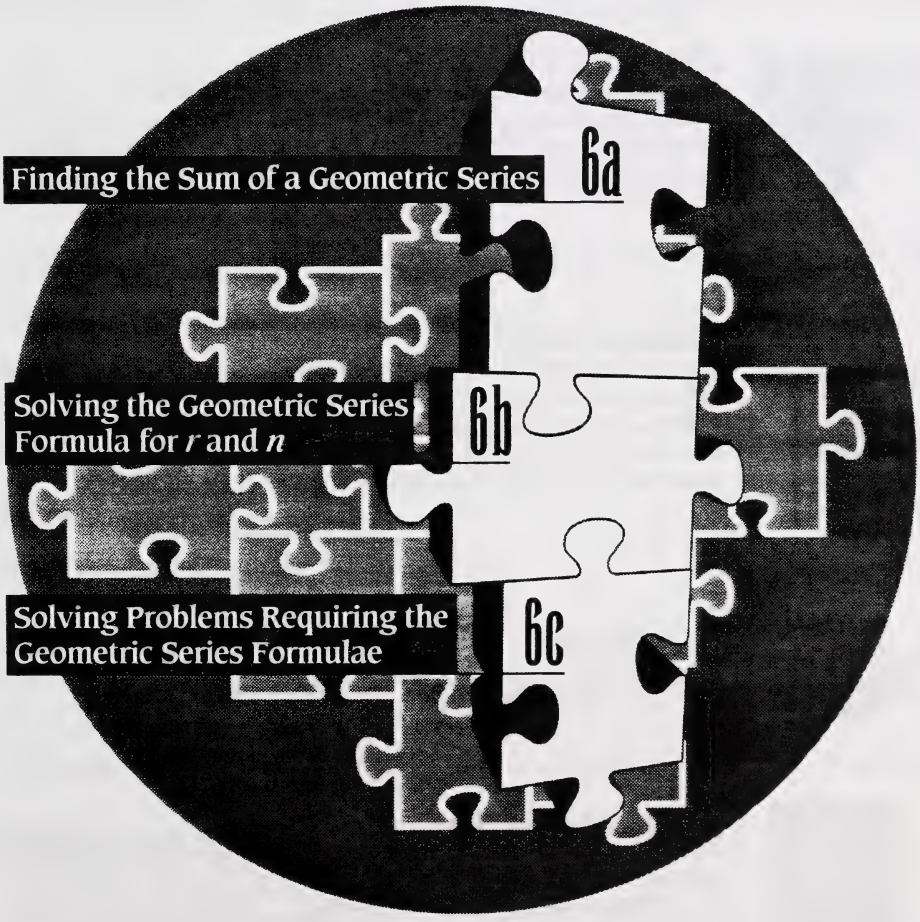
Lesson 6: GEOMETRIC SERIES

LEARNER EXPECTATIONS

By the end of Lesson 6 you should be able to

- demonstrate an understanding that a geometric series is the sum of the terms of a geometric sequence
- apply the sum formulae for a geometric series to solve problems

This lesson has three parts:


Finding the Sum of a Geometric Series
6a
Solving the Geometric Series Formula for r and n
6b
Solving Problems Requiring the Geometric Series Formulae
6c

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- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Finding the Sum of a Geometric Series

Watch the Introduction and Video Segment 6a.

Example 1

Using one of the formulae, find the sum of the geometric series 5 + 10 + 20 + 40 + 80.

Solution

When working with these types of questions a good general rule is to list all of the known information and then choose a formula based on that information.

$a = 5$, $r = 2$ and $n = 5$

$$S_n = \frac{a\left(r^n - 1\right)}{\left(r - 1\right)}$$

$$S_5 = \frac{5\left(2^5 - 1\right)}{\left(2 - 1\right)}$$

$$= 155$$

Example 2

Find the sum of the geometric series 2 + 4 + 8 + ... + 4096.

Solution

$a = 2$, $r = 2$ and $t_n = 4096$

$$S_n = \frac{\left(rt_n - a\right)}{r - 1}$$

$$= \frac{\left[2(4096) - 2\right]}{2 - 1}$$

$$= 8190$$

Additional Examples
Mathematics 12: p. 398 (Examples 1, 2)
Mathematics: Principles and Process 12: pp. 372, 373
Holtmath 12: p. 430 (Examples 1, 2)

Exercises

1. Show how the formulae can be used to find the sum of the following geometric series:

   a. $1 - 2 + 4 - 8 + 16 - 32 + 64 - 128 + 256 - 512 + 1024$

   b. $5 + 15 + 45 + \ldots + 98\,415$

2. In a geometric series, the first term is 1 and the common ratio is 3. If the nth term, t_n, is 6561, then correct to the nearest tenth determine the value of S_n.

3. Correct to the nearest tenth, determine the sum of the first seven terms of the geometric series $36 + 18 + 9 + \ldots$.

4. Find the sum of the first ten terms of the geometric series $5 + 10 + 20 + 40 + \ldots$.

EXERCISE 6a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 400 (Questions 1, 2, 4, 5, 10)
Mathematics: Principles and Process 12: p. 373 (Questions 1-3, 5)
Holtmath 12: p. 431 (Questions 3, 5, 6)

Solving the Geometric Series Formula for r and r

Find the common ratio of a geometric series where the first term is 2, the last term is 1458 and the sum is 2186.

Example 1

The first term of a geometric series is 3. The last term of the series is 384. The sum of the series is 765. Find the value of r.

Solution

List the information given, and then select an appropriate formula.

$a = 3,\ t_n = 384,\ S_n = 765$ and $r = ?$

$$S_n = \frac{(rt_n - a)}{r-1}$$

$$765 = \frac{[r(384) - 3]}{r-1}$$

$$765(r-1) = 384r - 3$$

$$765r - 765 = 384r - 3$$

$$381r = 762$$

$$r = 2$$

Example 2

The president of a major corporation was hired at a base salary of \$90 000 with yearly increases of 25%. How many complete years will it take for the president to have total earnings in excess of \$5 000 000?

Solution

$a = 90\,000$, $r = 1.25$, $S_n = 5\,000\,000$ and $n = ?$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$5\,000\,000 = \frac{90\,000(1.25^n - 1)}{0.25}$$

$$14.888\,888\,888\,9 = 1.25^n$$

$$\log(14.888\,888\,888\,9) = n \log 1.25$$

$$12.1 \doteq n$$

Note: Rounding should not be done until the last step.

It will take 13 years for the president to earn a total of at least \$5 000 000.

TEXT

Additional Examples
Holtmath 12: p. 431 (Examples 5, 6)

Exercises

5. A rubber ball is dropped from a height of 40 m. It rebounds on each bounce three-fifths of the distance from where it fell. Correct to the nearest tenth of a metre, how far will the ball have travelled by the time it hits the ground for the tenth time?

6. In an ultra-marathon, Bill ran the first ten kilometres in one hour. Each successive ten-kilometre block took Bill 25% longer. How far had he run after eight hours?

7. The seed pod of a certain plant contains four seeds. Next year each of these seeds will produce four more seeds. At this rate, how many seeds will there be at the end of 12 years?

EXERCISE 6b SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 401 (Questions 11, 13, 14)
Mathematics: Principles and Process 12: p. 374 (Questions 6, 11)
Holtmath 12: p. 431 (Questions 9, 10, 12, 13)

Solving Problems Requiring the Geometric Series Formulae

Find the first five terms in a geometric series if the first term is 7, the sum of five terms is 847 and the sum of six terms is 2548.

Watch Video Segment 6c.

Example 1

The seventh term in a geometric series is 256 and the first term is 4. Find the first four terms in the series.

Solution

$a = 4$ and $t_7 = 256$

$t_n = ar^{n-1}$

$256 = 4r^6$

$64 = r^6$

$r = \pm 2$

If $r = 2$, the first four terms in the series are $4 + 8 + 16 + 32$. If $r = -2$, the first four terms in the series are $4 - 8 + 16 - 32$.

Example 2

In a geometric series, the first term is 2, the sum of seven terms is 2186 and the sum of eight terms is 6560. Find the first three terms in the series.

Solution

$a = 2$, $S_7 = 2186$ and $S_8 = 6560$

$t_8 = S_8 - S_7$

$= 6560 - 2186$

$= 4374$

$t_n = ar^{n-1}$

$4374 = 2r^7$

$2187 = r^7$

$3 = r$

Use logarithms or the seventh root on your calculator to solve this.

The first three terms in the series are $2 + 6 + 18$.

Exercises

8. In a geometric series, the first term is -2, the sum of five terms is -62 and the sum of the first six terms is -126. Find the first five terms in the series.

9. In a geometric series, the first term is 3, the sum of the first four terms is 120 and the sum of the first five terms is 363. Find the first four terms in the series.

EXERCISE 6c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 401 (Questions 12-14)
Mathematics: Principles and Process 12: p. 374 (Questions 7, 8)
Holtmath 12: p. 431 (Questions 8, 11)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. Recognizing patterns is an important factor in deciding how to solve a particular problem. Many of these patterns are geometric in nature and such problem-solving situations often involve using the formula for finding the sum of a geometric series.

Solutions

EXERCISE 6a

1. a. $a = 1$, $r = -2$ and $t_n = 1024$

$$S_n = \frac{(rt_n - a)}{(r-1)}$$
$$= \frac{[(-2)(1024) - 1]}{(-2-1)}$$
$$= 683$$

b. $a = 5$, $r = 3$ and $t_n = 98\,415$

$$S_n = \frac{(rt_n - a)}{(r-1)}$$
$$= \frac{[(3)(98\,415) - 5]}{(3-1)}$$
$$= 147\,620$$

2. $a = 1$, $r = 3$ and $t_n = 6561$

$$S_n = \frac{(rt_n - a)}{(r-1)}$$
$$= \frac{[(3)(6561) - 1]}{3-1}$$
$$= 9841$$

3. $a = 36$, $r = 0.5$ and $n = 7$

$$S_n = \frac{a(r^n - 1)}{r-1}$$
$$S_7 = \frac{36(0.5^7 - 1)}{(0.5-1)}$$
$$\doteq 71.4$$

4. $a = 5$, $r = 2$ and $n = 10$

$$S_n = \frac{a(r^n - 1)}{r-1}$$
$$S_{10} = \frac{5(2^{10} - 1)}{(2-1)}$$
$$= 5115$$

EXERCISE 6b

5. $a = 40$, $n = 10$ and $r = 0.6$

total distance $= 2S_{10} - 40$

$$2S_{10} - 40 = \frac{80(0.6^{10} - 1)}{(0.6-1)} - 40$$
$$\doteq 158.79$$

The total distance travelled is approximately 158.8 m.

6. The terms of the series are the times taken to run each ten-kilometre block. Since each successive block took 25% longer, $r = 1.25$. The total number of hours is 8.

$a = 1$, $r = 1.25$ and $S_n = 8$

$$S_n = \frac{a(r^n - 1)}{r-1}$$
$$8 = \frac{1(1.25^n - 1)}{1.25-1}$$
$$2 = 1.25^n - 1$$
$$4.92 \doteq n$$

In eight hours, Bill had run about 4.92 ten-kilometre blocks for a total distance of about 49.2 km.

7. $a = 4$, $r = 4$ and $n = 12$

$$S_n = \frac{a(r^n - 1)}{r-1}$$
$$S_{12} = \frac{4(4^{12} - 1)}{(4-1)}$$
$$= 22\,369\,620$$

EXERCISE 6c

8. $a = -2$, $S_5 = -62$ and $S_6 = -126$

$t_6 = S_6 - S_5$

$t_6 = -64$

$t_n = ar^{n-1}$

$t_6 = (-2)r^5$

$-64 = (-2)r^5$

$32 = r^5$

$2 = r$

The first five terms in the series are $-2 -4 -8 -16 -32$.

9. $a = 3$, $S_4 = 120$ and $S_5 = 363$

$t_5 = S_5 - S_4$

$t_5 = 243$

$t_n = ar^{n-1}$

$t_5 = 3r^4$

$243 = 3r^4$

$81 = r^4$

$\pm 3 = r$

If $r = 3$, the first four terms in the series are 3, 9, 27 and 81. $S_4 = 120$; thus, this is a valid solution.

If $r = -3$, the first four terms in the series are $3 -9 +27 -81$. $S_4 = -60$. This does not fit the original question.

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MATH FACTOR

SEQUENCES AND SERIES

Lesson 7:
Sigma Notation

Student Workbook

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This guide accompanies **Sigma Notation** from the Math Factor series.

Writer: Chris Ream
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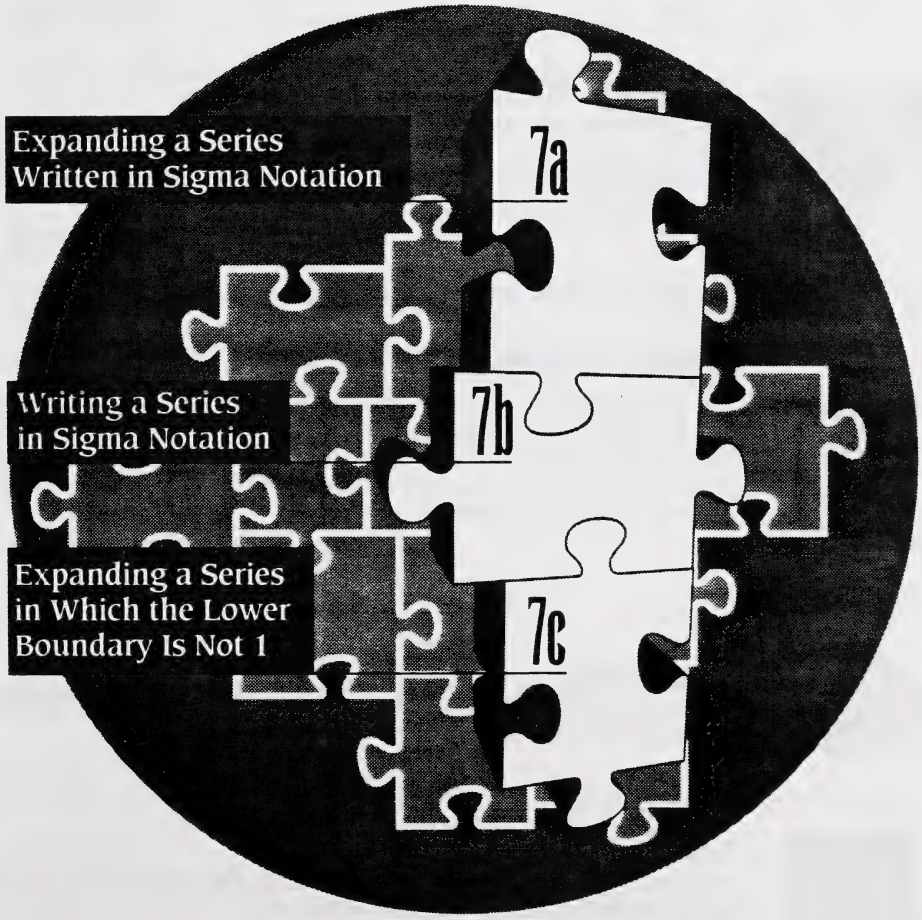
Lesson 7: SIGMA NOTATION

LEARNER EXPECTATIONS

By the end of Lesson 7 you should be able to

- write a series in sigma notation
- expand a series given in sigma notation

This lesson has three parts:


Expanding a Series Written in Sigma Notation
7a
Writing a Series in Sigma Notation
7b
Expanding a Series in Which the Lower Boundary Is Not 1
7c

EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Expanding a Series Written in Sigma Notation

Challenge yourself

Write out the series described by $\sum_{k=1}^{4}(4k+7)$.

MATH FACTOR

Watch the Introduction and Video Segment 7a.

Example 1

Expand the following series written in sigma notation.

$$\sum_{j=1}^{4} 3j$$

Solution

Recall that the $\sum_{j=1}^{4} 3j$ means that you are to create a series that has a general term of $3j$ and starts with $j = 1$ and ends with $j = 4$, where j is an integer.

$$\therefore \sum_{j=1}^{4} 3j = 3(1)+3(2)+3(3)+3(4)$$
$$= 3+6+9+12$$

Note that this series is arithmetic.

Example 2

Expand the following series written in sigma notation.

$$\sum_{a=1}^{6}\left(2^a+1\right)$$

Solution

Substitute values of a from 1 through 6 into the general term $\left(2^a+1\right)$.

$$\therefore \sum_{a=1}^{6}\left(2^a+1\right) = \left(2^1+1\right)+\left(2^2+1\right)+\left(2^3+1\right)+\left(2^4+1\right)+\left(2^5+1\right)+\left(2^6+1\right)$$
$$= 3+5+9+17+33+65$$

Note that this series is neither arithmetic nor geometric.

Additional Examples
Mathematics 12: p. 410 (Example 1)
Mathematics: Principles and Process 12: p. 364 (Example 1)
Holtmath 12: p. 432 (Example 1)

Exercises

1. Expand the following series written in sigma notation. State whether the series is arithmetic, geometric or neither.

a. $\sum_{k=1}^{6}(2k+1)$

b. $\sum_{k=1}^{6}(3k-1)$

c. $\sum_{n=1}^{5}2^n$

d. $\sum_{n=1}^{4}(-6)^{n-1}$

e. $\sum_{n=1}^{4}(xn+2)$

f. $\sum_{i=1}^{7}(2ai-3)$

g. $\sum_{j=1}^{5}(2^j-2)$

EXERCISE 7a SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 412 (Questions 1, 3, 6)
Mathematics: Principles and Process 12: p. 365 (Question 1)
Holtmath 12: p. 433 (Questions 3a, 3b, 4a, 4b, 4c, 4d, 7a, 7c, 7d)

Writing a Series in Sigma Notation

Challenge yourself

Write the arithmetic series $6+18+54+...+486$ in sigma notation.

MATH FACTOR

Watch Video Segment 7b.

Example

Write the following arithmetic series in sigma notation.

$$2+6+10+...+30$$

Solution

To write this series in sigma notation, you need to find the general term, t_n, and the number of terms, n.

Find the general term, t_n.

$$\begin{aligned} t_n &= a+(n-1)d \\ &= 2+(n-1)(4) \\ &= 2+4n-4 \\ &= 4n-2 \end{aligned}$$

Find the number of terms, n, by using the last term as t_n.

$$\begin{aligned} t_n &= 4n-2 \\ 30 &= 4n-2 \\ 32 &= 4n \\ 8 &= n \end{aligned}$$

Write the sigma notation using these calculated values.

$$\sum_{n=1}^{8}(4n-2)$$

Do a quick check by substituting in the values of n to make sure that you can duplicate the original series.

$$[4(1)-2]+[4(2)-2]+[4(3)-2]+...+[4(8)-2]=2+6+10+...+30$$

TEXT

Additional Examples
Mathematics 12: p. 410 (Example 2)
Mathematics: Principles and Process 12: p. 364 (Example 2)
Holtmath 12: p. 432 (Example 2)

Exercises

2. Write the following arithmetic series in sigma notation.

   a. $2 + 7 + 12 + 17 + 22$

   b. $30 + 25 + 20 + 15 + 10 + 5$

   c. $3 + 7 + 11 + \ldots + 283$

3. Write the following geometric series in sigma notation.

   a. $3+6+12+24+...+384$

   b. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...+\frac{1}{256}$

4. Write the following series, which are neither arithmetic nor geometric, in sigma notation.

   a. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+...+\frac{9}{10}$

   b. $\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+...+\frac{1}{30}$

5. In the Video Segment, a student was given the following series to write in sigma notation. The answer was not given. Write the series in sigma notation.

   $\frac{2}{3}+\frac{1}{2}+\frac{2}{5}+\frac{1}{3}+\frac{2}{7}+\frac{1}{4}$

EXERCISE 7b SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 412 (Questions 2, 7, 9, 12)
Mathematics: Principles and Process 12: p. 365 (Question 2)
Holtmath 12: p. 433 (Questions 5, 8)

Expanding a Series in Which the Lower Boundary Is Not 1

Example 1

Write the series defined by $\sum_{n=3}^{7}(2n-1)$.

Solution

This question has a different starting value for n $(n=3)$ than previous questions.

$$\sum_{n=3}^{7}(2n-1)=[2(3)-1]+[2(4)-1]+[2(5)-1]+[2(6)-1]+[2(7)-1]$$
$$=5+7+9+11+13$$

Note: There are **not** seven terms in the series. The series starts at $n=3$ and finishes at $n=7$; so, there are five terms in the solution.

In general, for the series $\sum_{k=a}^{n} f(k)$, there are $n-a+1$ terms.

For example, $\sum_{j=12}^{58}(2j-3)$ has $58-12+1=47$ terms.

Example 2

Determine the third term in the series defined by $\sum_{k=4}^{8} 2^k$.

Solution

The first term is when $k=4$.
The second term is when $k=5$.
The third term is when $k=6$.
So, $2^6 = 64$. The third term is 64.

Example 3

Determine the value of x in the following equation.

$$\sum_{i=1}^{3}(xi-2)=30$$

Solution

Substitute 1, 2 and 3 for i. This gives you three terms that will equal 30.

$$\begin{aligned}(x-2)+(2x-2)+(3x-2)&=30\\ 6x-6&=30\\ 6x&=36\\ x&=6\end{aligned}$$

Exercises

6. Write the series defined by $\sum_{n=10}^{20}(4n-5)$, and find its sum.

7. a. Write the series corresponding to the expression $\sum_{m=3}^{10}3m$.

   b. Write the same series in sigma notation, but let the lower limit be $m = 1$. Check your answer by writing out the series.

8. Determine the eighth term in the series defined by $\sum_{n=3}^{12}\left(3n^2-2\right)$.

9. Find the sum of the series $\sum_{k=1}^{3}2k+\sum_{k=3}^{7}\left(3k+2\right)$.

10. Determine the value of x in each of the following equations.

a. $\sum_{n=1}^{4}\left(xn+2\right)=28$

b. $\sum_{i=1}^{4}\left(xi-2i\right)=20$

EXERCISE 7c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 413 (Questions 10, 11)
Mathematics: Principles and Process 12: p. 365 (Questions 3e, 9)
Holtmath 12: p. 433 (Questions 2, 3b, 3d, 4e, 4f, 4g, 4h, 7b)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. Sigma notation is used in the Mathematics 30 Statistics unit and in other math and science courses.

Solutions

EXERCISE 7a

1. a. arithmetic $3+5+7+9+11+13$

   b. arithmetic $2+5+8+11+14+17$

   c. geometric $2+4+8+16+32$

   d. geometric $(-6)^0+(-6)^1+(-6)^2+(-6)^3=1-6+36-216$

   e. arithmetic $(x+2)+(2x+2)+(3x+2)+(4x+2)$

   f. arithmetic $(2a-3)+(4a-3)+(6a-3)+(8a-3)+(10a-3)+(12a-3)+(14a-3)$

   g. neither $(2^1-2)+(2^2-2)+(2^3-2)+(2^4-2)+(2^5-2)=0+2+6+14+30$

EXERCISE 7b

2. a. $a=2$, $d=5$ and $n=5$

$$t_n=a+(n-1)d$$
$$=2+(n-1)5$$
$$=5n-3$$
$$\sum_{n=1}^{5}(5n-3)$$

   b. $a=30$, $d=-5$ and $n=6$

$$t_n=a+(n-1)d$$
$$=30+(n-1)(-5)$$
$$=-5n+35$$
$$\sum_{n=1}^{6}(-5n+35)$$

   c. $a=3$, $d=4$ and $t_n=283$

$$t_n=a+(n-1)d$$
$$=3+(n-1)4$$
$$=4n-1$$
$$\sum_{n=1}^{71}(4n-1)$$

$$t_n=4n-1$$
$$283=4n-1$$
$$284=4n$$
$$71=n$$

3. a. $a=3$, $r=2$ and $t_n=384$

$$t_n=ar^{n-1}$$
$$=3(2)^{n-1}$$

$$t_n=3(2)^{n-1}$$
$$384=3(2)^{n-1}$$
$$128=(2)^{n-1}$$
$$2^7=(2)^{n-1}$$
$$7=n-1$$
$$8=n$$

$$\sum_{n=1}^{8}3(2)^{n-1}$$

   b. $a=1$, $r=\frac{1}{2}$ and $t_n=\frac{1}{256}$

$$t_n=ar^{n-1}$$
$$=1\left(\frac{1}{2}\right)^{n-1}$$

$$t_n=\left(\frac{1}{2}\right)^{n-1}$$
$$\frac{1}{256}=\left(\frac{1}{2}\right)^{n-1}$$
$$\left(\frac{1}{2}\right)^8=\left(\frac{1}{2}\right)^{n-1}$$
$$8=n-1$$
$$9=n$$

$$\sum_{n=1}^{9}\left(\frac{1}{2}\right)^{n-1}$$

4. a. Notice that the denominator is always one more than the numerator. If you let the numerator equal n, then the denominator will always be $n + 1$.

$$\sum_{n=1}^{9} \frac{n}{n+1}$$

b. Notice that the numerator is always 1 and the denominator is an arithmetic sequence (4, 5, 6, 7, ..., 30). Use the denominator to find t_n.

$a = 4$, $d = 1$ and $t_n = 30$

$$\begin{aligned} t_n &= a + (n-1)d \\ &= 4 + (n-1)1 \\ &= n + 3 \end{aligned}$$

$$\begin{aligned} t_n &= n + 3 \\ 30 &= n + 3 \\ 27 &= n \end{aligned}$$

$$\sum_{n=1}^{27} \frac{1}{n+3}$$

5. Rewrite the series as $\frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \frac{2}{6} + \frac{2}{7} + \frac{2}{8} = \sum_{k=1}^{6} \frac{2}{(k+2)}$.

EXERCISE 7c

6. $35 + 39 + 43 + 47 + 51 + 55 + 59 + 63 + 67 + 71 + 75$

The series is arithmetic.

$$\begin{aligned} \therefore S_n &= \frac{11}{2}[70 + 10(4)] \\ &= \frac{11}{2}(110) \\ &= 605 \end{aligned}$$

7. a. $9 + 12 + 15 + 18 + 21 + 24 + 27 + 30$

b. $\sum_{m=1}^{8} (3m + 6)$

8. The eighth term occurs when $n = 10$.

$$3(10)^2 - 2 = 298$$

9. $\sum_{k=1}^{3} 2k = 2 + 4 + 6$ $\qquad$ $\sum_{k=3}^{7} (3k + 2) = 11 + 14 + 17 + 20 + 23$

$= 12$ $\qquad$ $= 85$

$$\therefore \sum_{k=1}^{3} 2k + \sum_{k=3}^{7} (3k + 2) = 12 + 85$$

$$= 97$$

10. a. $(x - 2) + (2x - 4) + (3x - 6) + (4x - 8) = 20$

$$10x - 20 = 20$$

$$x = 4$$

b. $(x + 2) + (2x + 2) + (3x + 2) + (4x + 2) = 28$

$$10x + 8 = 28$$

$$x = 2$$

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MATH FACTOR

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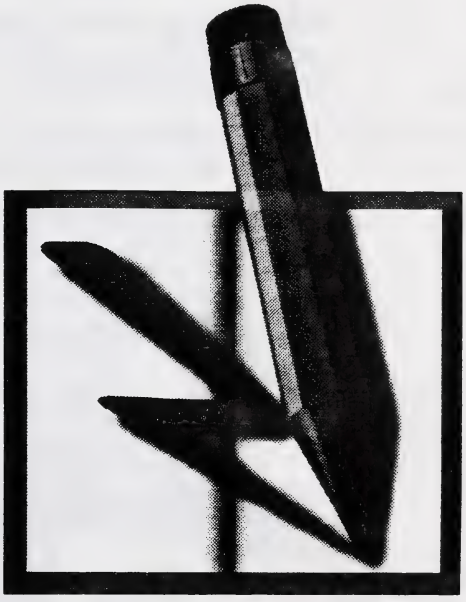

Lesson 8: Review

Student Workbook

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This guide accompanies **Review** from the Math Factor series.

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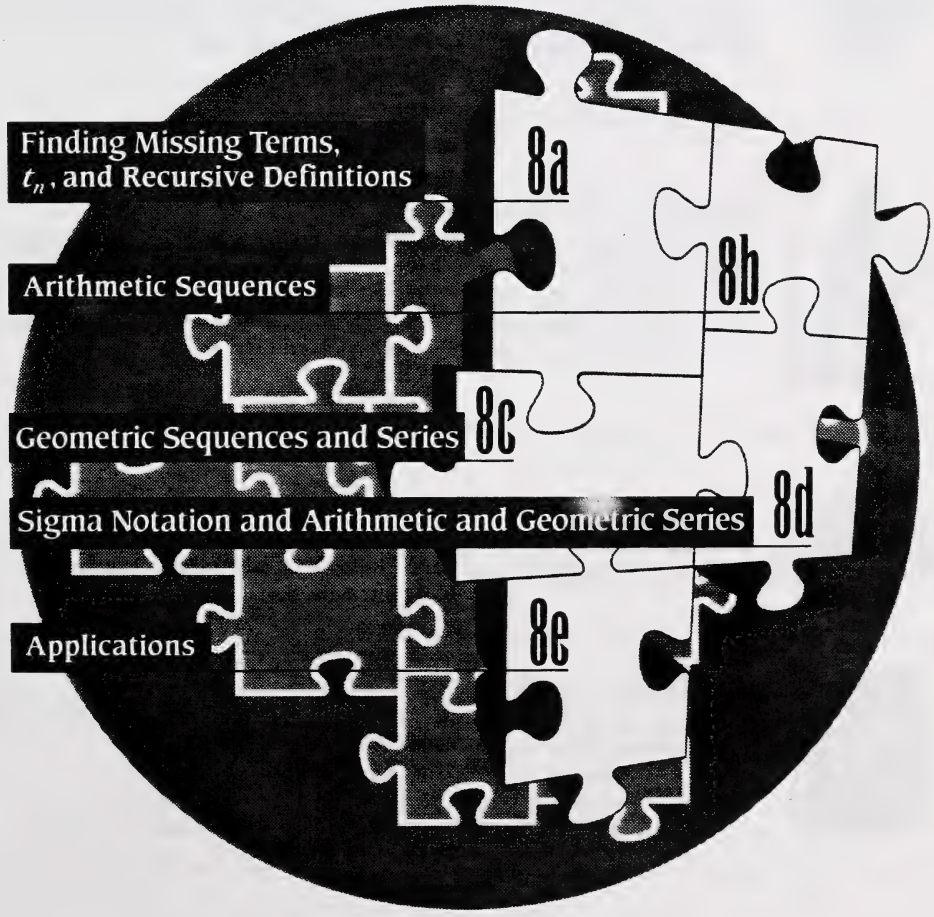
Lesson 8: REVIEW

LEARNER EXPECTATIONS

By the end of Lesson 8 you should be able to

- use a logical approach in determining the patterns observed in problems involving sequences and series
- apply the formulae for arithmetic and geometric sequences
- apply the formulae for arithmetic and geometric series
- solve a variety of problems by applying appropriate formulae or by using a logical approach

This lesson has five parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Finding Missing Terms, t_n and Recursive Definitions

Watch the Introduction and Video Segment 8a.

Example 1

Find the missing terms and write an expression for the general term, t_n, for the arithmetic sequence 2, 5, 8, ____, ____, ...

Solution

2, 5, 8, 11, 14, ...

$a = 2$ and $d = 3$

$$\begin{aligned} t_n &= a + (n-1)d \\ &= 2 + (n-1)(3) \\ &= 3n - 1 \end{aligned}$$

Example 2

Write the first four terms of the sequence defined recursively as $t_1 = 2$, $t_n = t_{n-1} + 3$ and $n > 1$.

Solution

$t_1 = 2$

$$\begin{aligned} t_2 &= t_{2-1} + 3 \\ &= t_1 + 3 \\ &= 5 \end{aligned}$$

$$\begin{aligned} t_3 &= t_{3-1} + 3 \\ &= t_2 + 3 \\ &= 8 \end{aligned}$$

$$\begin{aligned} t_4 &= t_{4-1} + 3 \\ &= t_3 + 3 \\ &= 11 \end{aligned}$$

Note that the next term, t_n, is found by adding 3 to the previous term, t_{n-1}.

Example 3

Find the next two terms and the general term of the sequence $\frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{7}{16}, \ldots$.

Solution

The sequence is neither arithmetic nor geometric. The numerator is arithmetic, however, and the denominator is geometric.

t_n for the numerator is $t_n = 1 + (n-1)2 = 2n - 1$, and t_n for the denominator is 2^n.

So, t_n for the entire sequence is $t_n = \frac{(2n-1)}{2^n}$, and $t_5 = \frac{9}{32}$ and $t_6 = \frac{11}{64}$.

Exercises

1. What is an infinite sequence?

2. What is meant by the general term of a sequence?

3. Write the missing terms and write the expression for the general term, t_n, for the following arithmetic sequences.

   a. 4, 10, 16, ___, ___, ...

   b. $4, -1, -6,$ ___, $-16,$ ___, ...

4. List the first three terms of the sequence defined by $t_n = -3n - 2$.

5. Write the first five terms of the sequence defined recursively by $t_1 = 1,\ t_2 = 2,\ t_n = t_{n-1} + t_{n-2} + 7,\ n > 2.$

6. Write a recursive definition for the geometric sequence $-2,\ 8,\ -32,\ 128,\ \ldots.$

7. A sequence is defined by $t_1 = -2,\ t_n = 2 - t_{n-1},\ n > 1$, where n is a natural number.

   a. Determine the first four terms of this sequence.

   b. Determine the 24th term of this sequence.

EXERCISE 8a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 427 (Questions 1-6, 20)
Mathematics: Principles and Process 12: p. 399 (Questions 2, 3), p. 400 (Questions 1, 2)
Holtmath 12: p. 455 (Question 1), p. 456 (Questions 1, 11)

Arithmetic Sequences

If three consecutive terms in a sequence are $x-3$, $2x-1$ and $4x-1$, then determine a value for x that will make the sequence arithmetic.

Example 1

Find the number of terms in the arithmetic sequence 2, 6, 10, ..., 202.

Solution

List the information known about the question.

$a=2,\ d=4,\ t_n=202$ and $n=?$

Choose a formula; then substitute and solve.

$$t_n = a+(n-1)d$$
$$202 = 2+(n-1)(4)$$
$$202 = 4n-2$$
$$204 = 4n$$
$$51 = n$$

There are 51 terms in this sequence.

Example 2

If three consecutive terms in a sequence are $2x-1$, $x+5$ and $3x+2$, then determine a value for x that will make the sequence arithmetic.

Solution

$2x-1,\ x+5,\ 3x+2$

When you subtract successive terms in an arithmetic sequence, you get the common difference. So, t_2-t_1 must equal t_3-t_2.

$$x+5-(2x-1) = 3x+2-(x+5)$$
$$x+5-2x+1 = 3x+2-x-5$$
$$-x+6 = 2x-3$$
$$9 = 3x$$
$$3 = x$$

Note the use of the brackets.

To check, substitute 3 for x into the sequence.

$2(3)-1,\ 3+5,\ 3(3)+2 = 5,\ 8,\ 11$

When $x = 3$, the three consecutive terms are 5, 8 and 11. These three terms are in an arithmetic sequence where the common difference is 3.

Exercises

8. How many terms are in the following arithmetic sequences.

   a. 32, 35, 38, ..., 65

   b. 163, 158, 153, ..., 43

   c. $\frac{2}{3}, \frac{6}{3}, \frac{10}{3}, \ldots, \frac{62}{3}$

   d. $2-2\sqrt{3},\ 2,\ 2+2\sqrt{3},\ \ldots,\ 2+44\sqrt{3}$

9. How many terms are there in the sequence $3^{100}, 3^{98}, 3^{96}, \ldots, 3^{12}$?

10. Determine a value of values for x so that x, $3x + 1$, $x^2 + 2$ are three consecutive terms in an arithmetic sequence.

11. Three consecutive terms of two different arithmetic sequences are given below. Find the three consecutive terms for each of the sequences.

    a. $3x+1,\ 4x,\ 4(x+2)$

    b. $x^2+2,\ 7x+1,\ 3x+10$

12. In an arithmetic sequence, the third term is 11 and the eighth term is 46. Find term t_{12}.

EXERCISE 8b SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 427 (Questions 7, 13, 16, 17)
Mathematics: Principles and Process 12: p. 399 (Question 4)
Holtmath 12: p. 455 (Question 3)

Geometric Sequences and Series

Challenge yourself

If $x-3$, $x+1$ and $4x-2$ are three consecutive terms in a geometric sequence, then find a value for x.

Watch Video Segment 8c.

Example

Given that the sum of n terms of a geometric series is given by $S_n = 2^n - 1$, find t_1, t_2 and t_3.

Solution

$S_n = 2^n - 1$

$S_1 = 2^1 - 1$

$= 1$

$\therefore t_1 = 1$

$S_2 = 2^2 - 1$

$= 3$

$t_2 = 2$ because $S_2 - S_1$ will give term 2.

$S_3 = 2^3 - 1$

$= 7$

$t_3 = 4$ because $S_3 - S_2$ will give term 3.

So, the first three terms of the sequence are 1, 2 and 4.

Exercises

13. If the sum of n terms in a geometric series is $S_n = 3^n - 1$, then find t_1, t_2 and t_3.

14. If $S_n = n(n+2)$ represents the sum of a geometric series, then determine the value of t_{10}.

15. Find values for y so that the following are consecutive terms in a geometric sequence.

    a. $y, y+3, 4y$

    b. $2y-4,\ 2y,\ 3y+4$

16. On a test the first and second terms of a sequence were x and y respectively. Student A assumed the sequence was arithmetic and calculated the common difference to be -2. Student B assumed the sequence was geometric and calculated the common ratio to be $\frac{2}{3}$. Find the values for x and y that are consistent with the assumptions of both students.

EXERCISE 8c SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 428 (Questions 23, 30)
Holtmath 12: p. 455 (Questions 4, 5, 16), p. 456 (Questions 3, 7, 12)

Sigma Notation and Arithmetic and Geometric Series

Given the series $20 + 14 + 8 + \ldots$, write an expression for S_n in terms of n; then find S_{20}.

Exercises

17. Find the sum for each of the following:

    a. $\sum_{k=1}^{6}(-2k+3)$

    b. $\sum_{k=3}^{7}(k^2+3)$

18. For each of the following arithmetic series, write an expression for S_n in terms of n; then find S_{10}.

    a. $3 + 6 + 9 + \ldots$

b. $32 + 28 + 24 + \ldots$

EXERCISE 8d SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 428 (Questions 24, 33)
Mathematics: Principles and Process 12: p. 400 (Question 5)
Holtmath 12: p. 455 (Question 9), p. 456 (Question 17)

Applications

$1500 is invested at 8% interest compounded annually. What would the value of the investment be after seven years?

Watch Video Segment 8e.

Additional Examples
Mathematics 12: pp. 300, 351
Holtmath 12: p. 434 (Examples 1, 3)

Exercises

19. $1000 is invested at an annual rate of 6% compounded annually. Correct to the nearest cent, what is the investment's total value after eight years?

20. During the first year after it was planted, a tree grew 60 cm. Each year thereafter, the yearly rate of growth was 60% of the previous year's growth. If the tree was originally 1.2 m high, what was its height at the end of ten years?

21. Write the next term of the sequence 9, 15, 23, 33.

EXERCISE 8e SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 354 (Questions 1-5)
Holtmath 12: p. 436 (Questions 4, 7, 11, 14, 15), p. 455 (#19, #20)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. It is important to understand and be able to apply the formulae you learned in this module. Science and real-life problems can be solved by observing patterns of numbers or shapes, which in turn are related to sequences and series.

Solutions

EXERCISE 8a

1. An infinite sequence is a sequence that has no last term.

2. The general term of a sequence is the nth term, t_n. It is a defining rule that can be used to describe any term in the entire sequence.

3. a. 22, 28, ...

$$\begin{aligned} t_n &= a+(n-1)d \\ &= 4+(n-1)6 \\ &= 6n-2 \end{aligned}$$

b. $-11, -16, -21, \ldots$

$$\begin{aligned} t_n &= a+(n-1)d \\ &= 4+(n-1)(-5) \\ &= -5n+9 \end{aligned}$$

4. $-5, -8, 11$

5. 1, 2, 10, 19, 36

6. $t_1 = -2,\ t_{n+1} = -4t_n,\ n>1$

or

$t_1 = -2,\ t_n = -4t_{n-1},\ n>1$

7. a. $-2,\ 4,\ -2,\ 4$

b. $t_{24} = 4$

EXERCISE 8b

8. a. $a = 32,\ d = 3$ and $t_n = 65$

$$\begin{aligned} t_n &= a+(n-1)d \\ 65 &= 32+(n-1)3 \\ n &= 12 \end{aligned}$$

b. $a = 163,\ d = -5$ and $t_n = 43$

$$\begin{aligned} t_n &= a+(n-1)d \\ 43 &= 163+(n-1)(-5) \\ n &= 25 \end{aligned}$$

c. $a = \frac{2}{3},\ d = \frac{4}{3}$ and $t_n = \frac{62}{3}$

$$\begin{aligned} t_n &= a+(n-1)d \\ \frac{62}{3} &= \frac{2}{3}+(n-1)\left(\frac{4}{3}\right) \\ n &= 16 \end{aligned}$$

d. $a = 2-2\sqrt{3},\ d = 2\sqrt{3}$ and $t_n = 2+44\sqrt{3}$

$$\begin{aligned} 2+44\sqrt{3} &= 2-2\sqrt{3}+(n-1)2\sqrt{3} \\ n &= 24 \end{aligned}$$

9. 100, 98, 96, ..., 12

$a = 100,\ d = -2$ and $t_n = 12$

$$\begin{aligned} 12 &= 100+(n-1)(-2) \\ n &= 45 \end{aligned}$$

10. $3x+1-x = x^2+2-(3x+1)$

$$\begin{aligned} x^2-5x &= 0 \\ x(x-5) &= 0 \end{aligned}$$

$x = 0$ or $x = 5$

11. a. $4x-(3x+1) = 4(x+2)-4x$

$$\begin{aligned} 4x-3x-1 &= 4x+8-4x \\ x &= 9 \end{aligned}$$

The three terms are 28, 36, 44.

b. $7x+1-(x^2+2) = 3x+10-(7x+1)$

$$\begin{aligned} -x^2+7x-1 &= -4x+9 \\ x^2-11x+10 &= 0 \end{aligned}$$

$x = 10$ or $x = 1$

The two possible answers are 102, 71, 40 or 3, 8, 13.

12. $t_3 = 11$, $t_8 = 46$ and $t_{12} = ?$

$t_n = a + (n-1)d$ $\qquad$ $t_n = a + (n-1)d$

$11 = a + (3-1)d$ $\qquad$ $46 = a + (8-1)d$

$11 = a + 2d$ ❶ $\qquad$ $46 = a + 7d$ ❷

Subtract ❶ from ❷.

$$\begin{aligned} 46 &= a + 7d \\ \underline{11} &= \underline{a + 2d} \\ 35 &= 5d \\ 7 &= d \end{aligned}$$

Substitute $d = 7$ into ❶ to solve for a.

$$\begin{aligned} 11 &= a + 2(7) \\ a &= -3 \end{aligned}$$

$$\begin{aligned} t_{12} &= -3 + (12-1)7 \\ &= -3 + (11)(7) \\ &= 74 \end{aligned}$$

EXERCISE 8c

13. $S_1 = 2$, $S_2 = 8$ and $S_3 = 26$ $\qquad$ $t_1 = 2$, $t_2 = 6$ and $t_3 = 18$

14. $S_9 = 99$ and $S_{10} = 120$ $\qquad$ $t_{10} = 120 - 99$
$= 21$

15. a. $\dfrac{(y+3)}{y} = \dfrac{4y}{(y+3)}$

$$\begin{aligned} y^2 + 6y + 9 &= 4y^2 \\ 3y^2 - 6y - 9 &= 0 \\ y^2 - 2y - 3 &= 0 \\ (y-3)(y+1) &= 0 \\ y = 3 \text{ or } y &= -1 \end{aligned}$$

Hence, the three terms are 3, 6, 12 or -1, 2, -4.

b. $\dfrac{2y}{(2y-4)} = \dfrac{(3y+4)}{2y}$

$$\begin{aligned} 4y^2 &= 6y^2 - 4y - 16 \\ y^2 - 2y - 8 &= 0 \\ (y-4)(y+2) &= 0 \\ y = 4 \text{ or } y &= -2 \end{aligned}$$

Hence, the three terms are 4, 8, 16 or -8, -4, -2.

16. The sequence is x, y. $\qquad$ The common difference is $y - x = -2$.

The common ratio is $\frac{y}{x} = \frac{2}{3}$ or $y = \frac{2x}{3}$.

Substitute $y = \frac{2x}{3}$ into $y - x = -2$ and solve.

$$\begin{aligned} \frac{2x}{3} - x &= -2 \\ -\frac{1}{3}x &= -2 \\ x = 6 \text{ and } y &= 4 \end{aligned}$$

EXERCISE 8d

17. a. $\sum_{k=1}^{6}(-2k+3) = (-2+3)+(-4+3)+(-6+3)+(-8+3)+(-10+3)+(-12+3)$

$= 1-1-3-5-7-9$

$= -24$

b. $\sum_{k=3}^{7}(k^2+3) = (9+3)+(16+3)+(25+3)+(36+3)+(49+3)$

$= 12+19+28+39+52$

$= 150$

18. a. $S_n = \dfrac{n[2a+(n-1)d]}{2}$ $\qquad S_{10} = \dfrac{10(33)}{2}$

$= \dfrac{n[6+(n-1)(3)]}{2}$ $\qquad = 165$

$= \dfrac{n(3n+3)}{2}$

b. $S_n = \dfrac{n[2a+(n-1)d]}{2}$ $\qquad S_{10} = -200+340$

$= \dfrac{n[64+(n-1)(-4)]}{2}$ $\qquad = 140$

$= \dfrac{n(-4n+68)}{2}$

$= -2n^2+34n$

EXERCISE 8e

19. At the end of year 1, $A = 1000(1.06) = 1060$; at the end of year 2, $A = 1060(1.06) = 1123.60$; and so on.

The amounts form a geometric sequence with $a = 1000$, $r = 1.06$ and $n = 8$.

$A = 1000(1.06)^8$

$= \$1593.85$ (to the nearest cent)

20. Growth in successive years is $0.6+0.6(0.6)+0.6(0.6)^2+\ldots$.

$S_{10} = \dfrac{0.6(0.6^{10}-1)}{(0.6-1)}$

$= 1.5$ (rounded to the nearest tenth)

Therefore, the tree will have a height of approximately 1.5 + 1.2 = 2.7 m.

21. The differences between successive terms are 6, 8, 10, ... If this pattern continues, the next difference will be 12; thus, the next term will be 33 + 12 = 45.